stab.routh Routh-Hurwitz criterion

There is no practical way to find the roots of a polynomial greater than degree four.[7] An implication of this is that we cannot practically solve (analytically) for the poles of a closed-loop transfer function with degree greater than four. Fortunately, **numerical root finders** can handle these higher-order systems with ease. However, there is a drawback to using numerical root finders to determine stability: design parameters, which show up in the coefficients of the denominator polynomial of a transfer function, must be assigned a specific value.

7. For the interested reader, see this stackexchange discussion.

numerical root finders

A couple of mathematicians[8] in the late 19th century came up with a clever test—called the **Routh-Hurwitz stability criterion**[9]—for learning much about the stability of a system without computing its poles; moreover, the test yields an analytically tractable way to determine ranges over which design parameters yield stable closed-loop systems.

8. Edward John Routh and Adolf Hurwitz were their names.

Routh-Hurwitz stability criterion

9. It is noteworthy that the criterion is based on the Routh-Hurwitz theorem.

An algorithm for applying the Routh-Hurwitz criterion

We consider an algorithm for this test. First, we address the "basic" algorithm and refer the reader to Nise[10] for the two exceptions that arise when Column 1 has a zero or when an entire row is zero. You can teach this algorithm (including the exceptions) to a computer, as some have, but it is easy enough by-hand for many systems.

10. Nise, *Control Systems Engineering, 7th Edition*.

Let the denominator of a closed-loop transfer function, with real coefficients a_i be

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n,$$

where n a finite integer greater than or equal to the order of the numerator polynomial and $a_0 > 0$ (if it is not, make it so by multiplication by -1). Perform the following two steps.

Table routh.1: the general form of the Routh table. Empty cells are always zero.

	1	2	3	4	$\cdots$	
s^n	a_0	a_2	a_4	a_6	$\cdots$	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$	$\cdots$
s^{n-4}	d_1	d_2	d_3	d_4	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^2	e_1	e_2				
s^1	f_1					
s^0	g_1					

Routh table

First, construct a **Routh table**. The procedure is to fill in the general form of the Routh table, shown in Table routh.1, with the definitions:

$$\begin{aligned}
b_1 &= -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}, & b_2 &= -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}, & b_3 &= -\frac{1}{a_1}\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}, & \cdots \\
c_1 &= -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}, & c_2 &= -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}, & c_3 &= -\frac{1}{b_1}\begin{vmatrix} a_1 & a_7 \\ b_1 & b_4 \end{vmatrix}, & \cdots \\
d_1 &= -\frac{1}{c_1}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}, & d_2 &= -\frac{1}{c_1}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}, & d_3 &= -\frac{1}{c_1}\begin{vmatrix} b_1 & b_4 \\ c_1 & c_4 \end{vmatrix}, & \cdots \\
&\vdots & &\vdots & &\vdots \\
g_1 &= -\frac{1}{f_1}\begin{vmatrix} e_1 & e_2 \\ f_1 & 0 \end{vmatrix}, & g_2 &= -\frac{1}{f_1}\begin{vmatrix} e_1 & 0 \\ f_1 & 0 \end{vmatrix}, & g_3 &= -\frac{1}{f_1}\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}.
\end{aligned} \tag{1}$$

Note the pattern that emerges in Equation 1. The number of rows and potentially nonzero columns are $n+1$ and $\lceil (n+1)/2 \rceil$. Potentially nonzero values hug Column 1. Descending rows, the number of potentially nonzero coefficients decreases.

basic Routh table interpretation

The second step is to **interpret** the Routh table. For the basic Routh table, no poles lie on the imaginary axis (which excludes marginal stability), so interpretation is simple: *the number of sign changes in Column 1 is equal to the number of poles in the right half-plane—and all others are in the left half-plane*. Therefore, the system is strictly stable if its Routh array is of the basic type and has no sign changes in Column 1.

Example stab.routh-1

re: Basic Routh table with an unknown parameter

Given the closed-loop transfer function

$$\frac{s+7}{s^3+3s^2+s+k} \tag{2}$$

where k is a design parameter, using the Routh-Hurwitz criterion, find the range of k for which the closed-loop system is stable.

Let's build the Routh table in Table routh.2. The lower entries were computed from

Equation 1 (n.b. we knew $b_2 = 0$, but compute it for demonstrative purposes) as follows:

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix} = -\frac{1}{\underline{\quad}}\begin{vmatrix} \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \end{vmatrix} = \underline{\qquad\qquad},$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix} = -\frac{1}{\underline{\quad}}\begin{vmatrix} \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} \end{vmatrix} = \underline{\quad}, \text{ and}$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix} = -\frac{1}{\underline{\qquad\qquad}}\begin{vmatrix} \underline{\quad} & \underline{\quad} \\ \underline{\qquad\qquad} & \underline{\quad} \end{vmatrix} = \underline{\quad}.$$

Table routh.2: Routh table for Example stab.routh-1.

	1	2	3			1	2	3
s^3	__	__	0		s^3	__	__	0
s^2	__	__	0	$\rightarrow$	s^2	__	__	0
s^1	__	__	0		s^1	________	__	0
s^0	__	0	0		s^0	_	0	0

Now we must interpret the result. Since the first two entries in Column 1 are positive, the last two must be in order for the system stability. The conditions are:

$$\underline{\qquad\qquad} > 0 \;\Rightarrow\; \underline{\qquad\quad} \text{ and}$$

$$k > \underline{\quad}.$$

Therefore, the range for stability is ________. Expressed as an interval, $k \in$ ______.