

## rldesign.PD Proportional-derivative (PD) controller design

Thus far, our designs have been restricted to closed-loop pole locations on the original root locus. We could add integral or lag compensation for steady-state error performance and vary the gain for transient response performance. But what if we desire closed-loop poles  $p_{1,2}$  to be in a location that the root locus does not intersect?

Among many possible methods to address this, we pursue the following: a **derivative compensator** with zero location  $z_c$  chosen such that the root locus intersects  $p_{1,2}$ , with form

$$K(s - z_c), \quad (1)$$

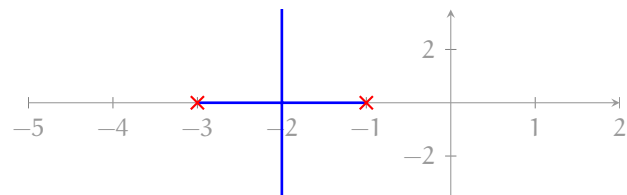
where  $K \in \mathbb{R}$  is a gain. This compensator is called “derivative” because its primary effect on the overall controller’s operation on the error  $e$  is a new factor of  $s$ , yielding a scaling of the term  $sE(s) = \dot{e}(t)$ .

The effect of this zero is to pull the locus toward it. Consider the simple plant of Fig. PD.1.

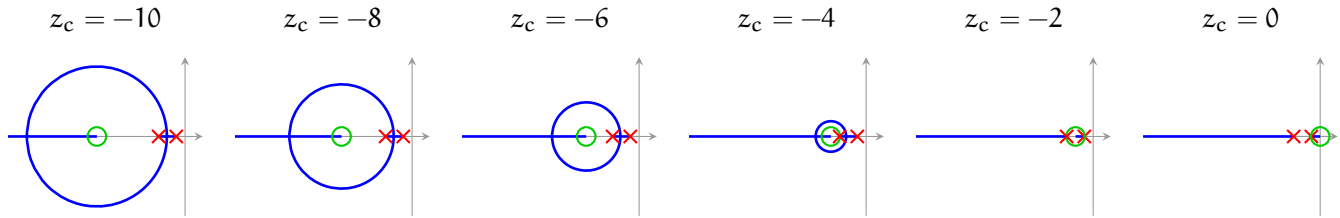
Suppose we would like to speed up the closed-loop response, but cannot because, no matter how much gain we use, the settling time is fixed by the vertical asymptotes. If we use a compensator zero at  $z_c$ , we can pull the locus leftward, as shown in Fig. PD.2. Varying  $z_c$  from  $-\infty$  to  $0$ , we see that any location left of  $-2$  can be intersected. In fact, if we consider both positive and negative gains for this example, we can place a desired closed-loop pole at any location in the complex plane!

A way to approach designing a controller for a plant  $G$  with a derivative compensator  $C$  is to consider the compensator zero’s effect on the phase criterion, which must always be satisfied

### derivative compensator



**Figure PD.1:** root locus for a simple plant with two poles.



**Figure PD.2:** root locus (blue) for plant with poles (red) compensated with a zero (green) at  $z_c$ . Note that varying  $z_c$  yields root loci that can intersect any point in the complex plane if negative gains are considered. An animation corresponding to this figure can be found at [https://youtu.be/VZbT\\_2bT2xU](https://youtu.be/VZbT_2bT2xU).

at points on the root locus:

$$\angle(G(s)C(s)) = \pi. \tag{2}$$

In order for a desired point  $s = \psi$  to be on the root locus, then,<sup>3</sup>

3. The  $2\pi$  modulo in these expressions is suppressed for clarity.



Let this angle  $\angle(\psi - z_c)$ , called the **compensator angle**, be given the symbol

$$\theta_c \equiv \angle(\psi - z_c). \tag{3}$$

Then

$$z_c = \text{Re}(\psi) - \text{Im}(\psi) / \tan \theta_c \quad (\theta_c \in [-\pi, \pi]), \tag{4}$$

where we have limited the application of this result to  $\theta_c \in [-\pi, \pi]$  because a single zero can contribute angles in this interval only.<sup>4,5</sup> This result is to be used in the design procedure that follows. It can be understood geometrically as the position of  $z_c$  such that the angle of the vector with tail at  $z_c$  and head at  $\psi$  is  $\theta_c$ .

4. See [Lec. rldesign.mutld](#) for how to handle required angle compensations beyond  $\pm\pi$ .

5. Note that  $\theta_c \in [-\pi, 0)$  is possible only when  $\text{Im } \psi < 0$  and  $\theta_c \in (0, \pi]$  is possible only when  $\text{Im } \psi > 0$ .

### Design procedure

The following procedure provides a starting-point for proportional-derivative

controller design. Let's assume the transient response specification is such that we desire a closed-loop pole to be located at  $s = \psi$ .

1. Design a proportional controller to meet transient response requirements by choosing the gain  $K_1$  for the dominant closed-loop poles to be as close as possible to  $\psi$ .
2. Include a cascade derivative compensator of the form

$$K_2(s - z_c), \quad (5)$$

where, initially,  $K_2 = 1$  and  $z_c$  is a real zero that satisfies Eq. 4. For convenience, we repeat the two key formulas:

$$\theta_c = \pi - \angle G(\psi) \quad \text{and}$$

$$z_c = \text{Re}(\psi) - \text{Im}(\psi) / \tan \theta_c \quad (\theta_c \in [-\pi, \pi]).$$

3. Use a new root locus to tune the gain  $K_2$  such that a closed-loop pole is at  $\psi$ .
4. Construct the closed-loop transfer function with the controller

$$K_1 K_2 (s - z_c). \quad (6)$$

5. Simulate the time response to see if it meets specifications. Tune.

## A design example

Let a system have plant transfer function

$$\frac{1}{(s + 2)(s + 6)(s + 11)}. \quad (7)$$

Design a PD controller such that the closed-loop settling time is about 0.8 seconds and the overshoot is about 15%.

### Determining $\psi$

We use Matlab for the design.<sup>6</sup> First, we must determine what the specified transient response

6. See [ricopic.one/control/source/pd\\_controller\\_design\\_example.m](https://ricopic.one/control/source/pd_controller_design_example.m) for the source.

criteria imply for the locations of our closed-loop poles. Let one of these desired pole locations be called  $\psi$ . The transient response performance criteria are as follows.

```
Ts = .8; % sec ... spec settling time
OS = 15; % percent ... spec overshoot
```

The second-order approximation from [Chapter trans](#) tells us that the settling time specification implies a specific  $\text{Re}(\psi)$  and the overshoot a specific angle  $\angle\psi$ . The real part is found from the expressions

$$T_s = \frac{4}{\zeta\omega_n} \quad \text{and} \quad \text{Re}(\psi) = -\zeta\omega_n \Rightarrow \quad (8)$$

$$\text{Re}(\psi) = -\frac{4}{T_s}. \quad (9)$$

The angle is found via the equations

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}, \quad (10)$$

$$\tan(\angle\psi) = \frac{\sqrt{1-\zeta^2}}{\zeta}, \quad \text{and} \quad \tan(\angle\psi) = -\text{Im}(\psi)/\text{Re}(\psi). \quad (11)$$

A remarkably simple expression results:

$$\text{Im}(\psi) = -\text{Re}(\psi) \frac{\sqrt{1-\zeta^2}}{\zeta} \quad (12a)$$

$$\text{Im}(\psi) = -\text{Re}(\psi) \frac{\pi}{\ln(100/\%OS)}. \quad (12b)$$

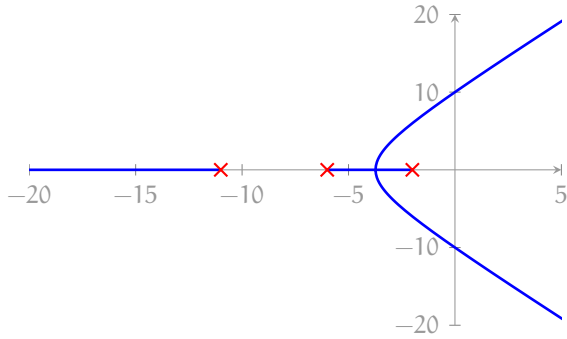
So, in the final analysis, the desired pole location  $\psi$  (assuming the second-order approximation is valid) is given by the expression

$$\psi = -\frac{4}{T_s} \left( 1 - j \frac{\pi}{\ln(100/\%OS)} \right). \quad (13)$$

This formula holds beyond the scope of this problem. We define it as an anonymous function.

```
psi_fun = @(Ts,pOS) -4/Ts*(1-1j*pi/log(100/pOS));
psi = psi_fun(Ts,OS);
disp(sprintf('psi = %0.3g + j %0.3g',real(psi),imag(psi)))
```

psi = -5 + j 8.28



**Figure PD.3:** root locus without compensation.

*P control*

We design a proportional controller that gets us as close as possible to  $\psi$ . The root locus is shown in [Figure PD.3](#).

```
G = zpk([], [-2, -6, -11], 1);
figure
rlocus(G)
```

Although we cannot get close to  $\psi$  on the root locus, we can at least meet our %OS specification by choosing a gain of about

$$K_1 = 240. \tag{14}$$

Let's construct the compensator and corresponding closed-loop transfer function  $G_P$  for gain control.

```
K1 = 240;
G_P = feedback(K1*G, 1);
```

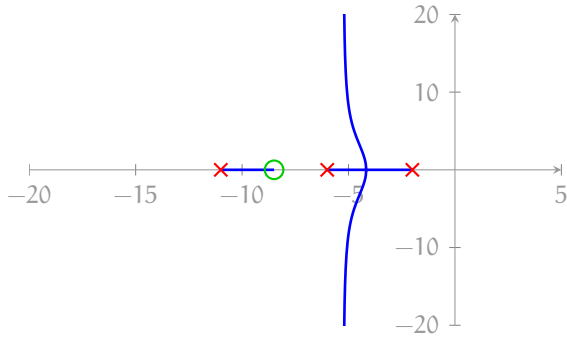
*Derivative compensation*

Now, we use cascade derivative compensation with compensator

$$K_2(s - z_c). \tag{15}$$

For now, we set  $K_2 = 1$ . From [Equation 4](#), we compute the compensator zero

$$z_c = \text{Re}(\psi) - |\text{Im}(\psi)| / \tan \theta_c \quad \text{and} \quad \theta_c = \pi - \angle G(\psi).$$



**Figure PD.4:** root locus with compensation.

```
theta_c = pi - angle(evalfr(G,psi));
z_c = real(psi) - abs(imag(psi))/tan(theta_c);
disp(sprintf('theta_c = %0.3g deg',rad2deg(theta_c)))
disp(sprintf('z_c = %0.3g',z_c))
```

```
theta_c = 67.1 deg
z_c = -8.5
```

Let’s construct the compensator sans tuned gain  $K_2$  and tune it up using another root locus.

```
C_sans = zpk(z_c, [], 1);
figure
rlocus(K1*C_sans*G)
```

The resulting root locus of **Figure PD.4** intersects  $\psi$ ! (I mean, we knew it would, but we had our doubts.) The corresponding gain is, from **Equation 2** (or we could use the data cursor),

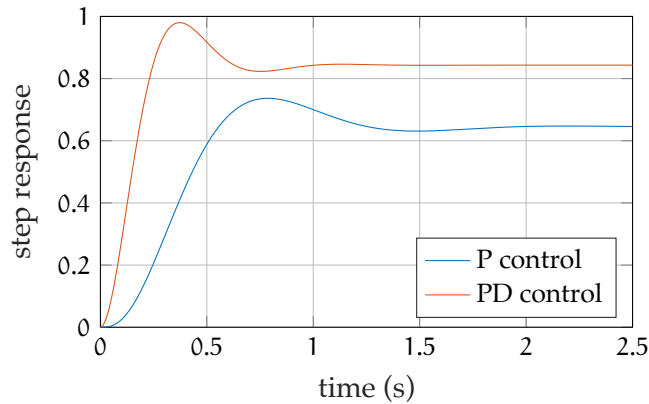
$$K_2 = \frac{1}{|(\psi - z_c)G(\psi)|}. \tag{16}$$

Let’s compute it, the controller  $C_{PD}$ , and the closed-loop transfer function  $G_{PD}$ .

```
K2 = 1/abs(evalfr(K1*C_sans*G,psi));
C = K1*K2*C_sans;
G_PD = feedback(C*G,1);
```

*Simulate*

Our placement of the  $\psi$  depended on the second-order approximation’s accuracy, which



**Figure PD.5:** step responses for proportional and proportional-derivative controllers.

in this case is questionable, due to the proximity of a third closed-loop pole. In any case, we simulate the step response to test the efficacy of the PD controller design and to compare it with the P controller.

```
t_a = linspace(0,2.5,200); % s ... sim time
y_P = step(G_P,t_a); % P controlled step response
y_PD = step(G_PD,t_a); % PD controlled step response
```

```
figure
plot(t_a,y_P);
hold on;
plot(t_a,y_PD);
xlabel('time (s)');
ylabel('step response');
grid on
legend('P control','PD control','location','southeast');
```

The responses, shown in [Figure P.Lag.3](#), suggest the PD controller is at least close to meeting the transient specifications. It is a happy accident that the steady-state error also improved; derivative compensation does not always do this. Let's use `stepinfo` to compute more accurate transient response characteristics of the PD-controlled system.

```
si_PD = stepinfo(y_PD,t_a);  
disp(sprintf('settling time: %0.3g',si_PD.SettlingTime))  
disp(sprintf('percent overshoot: %0.3g',si_PD.Overshoot))
```

```
settling time: 0.82  
percent overshoot: 16.2
```

This is quite close to the specification. If desired, the gain  $K_2$  and the zero location  $z_c$  could be tuned, iteratively.