ss.sfdbck Controller design method

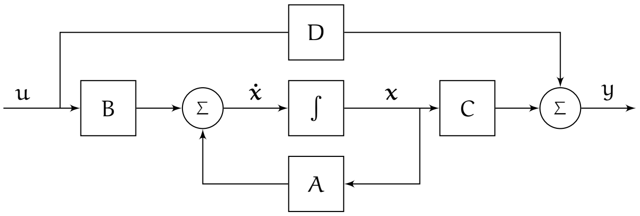

Figure sfdbck.1: the plant state model of Eq. 1 written in block diagram form.

We will consider single-input single-output (SISO) control plants that can be written with input u; state vector $\mathbf{x}$; output y; state model matrices A, B, C, and D; and state and output equations

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \tag{1a}$$

$$y = C\mathbf{x} + Du. \tag{1b}$$

Plants of this form can be written in block diagram form, as illustrated in Fig. sfdbck.1. In general, SISO systems are of order n with n state variables.

Let us consider the following feedback control method called **state feedback control**. We will feed back the state vector $\mathbf{x}$, operate on it with a $1 \times n$ vector of gains $\mathbf{K} \in \mathbb{R}^n$, and subtract the result from the command r, the result of which becomes the input u, as shown in Fig. sfdbck.2.

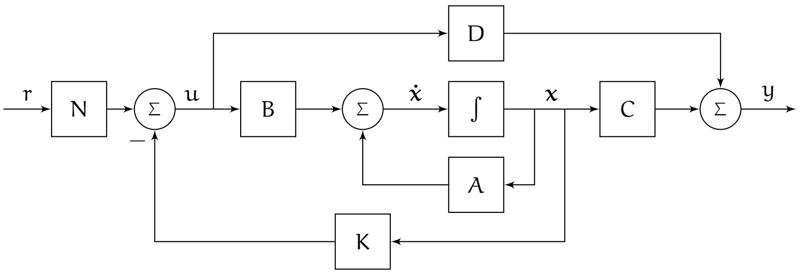

Figure sfdbck.2: the state feedback control block diagram.

state feedback control

The control problem for state feedback control is to determine the n gains in $\mathbf{K}$ such that the closed-loop poles are located in desirable positions. The gain $N \in \mathbb{R}$ is provided for steady-state error considerations, which will be addressed in Lec. ss.sfdbck. A new state model can be derived for the closed-loop system as follows. Let us consider the command r to be our new "input," instead of u, which is now the control effort. From the block diagram,

$$u = Nr - K\mathbf{x}, \tag{2}$$

which can be substituted into Eq. 1 to define the new state model

$$\dot{\mathbf{x}} = (A - BK)\mathbf{x} + NBr \tag{3a}$$

$$y = (C - DK)\mathbf{x} + NDr. \tag{3b}$$

The eigenvalues of $A - BK$, which can be found from equating zero and the **closed-loop**

closed-loop characteristic polynomial

characteristic polynomial

$$P_K = \det(sI - A + BK), \tag{4}$$

are equal to the closed-loop poles, which we would like to place in specific locations. Those specific locations can be specified by the **design characteristic polynomial** P_d. P_K depends on the n gains K_i, and n equations can be found by equating the polynomial coefficients of P_K and P_d.

design characteristic polynomial

Solving for K_i is straightforward but can be very tedious in the general case. Let the coefficients of P_d be δ_i and those of P_K be denoted κ_i. Then the $n \times 1$ vector containing κ_i can be expressed as a linear combination of K_i as

$$\boldsymbol{\kappa} = \mathcal{K}\,\mathbf{K}^\top, \tag{5}$$

where $\mathcal{K}$ is an $n \times n$ matrix of coefficients that were derived from A and B. Let $\boldsymbol{\delta}$ be the $n \times 1$ vector of components δ_i. Since the vector $\boldsymbol{\delta}$ is specified by our design requirements, we can solve for $\mathbf{K}$ as follows.

$$\boldsymbol{\kappa} = \boldsymbol{\delta}, \tag{6}$$

and therefore,

$$\begin{aligned} \mathcal{K}\,\mathbf{K}^\top &= \boldsymbol{\delta} \quad \Longrightarrow \\ \mathbf{K}^\top &= \mathcal{K}^{-1}\,\boldsymbol{\delta} \quad \Longrightarrow \\ \mathbf{K} &= \left(\mathcal{K}^{-1}\,\boldsymbol{\delta}\right)^\top. \end{aligned} \tag{7}$$

Eq. 7 is valid for all cases in which $\mathcal{K}$ is invertible.[1] However, there is a special form of the original state-space model that always yields a simple solution for $\mathbf{K}$: the **phase-variable canonical form** (see Appendix B.02).

1. We leave the following as an open question: under what conditions is $\mathcal{K}$ invertible?

phase-variable canonical form

Solving for the gain via the phase-variable canonical form

The phase-variable canonical form of the original system is:

$$\dot{\mathbf{x}}_c = A_c \mathbf{x}_c + B_c u \tag{8a}$$

$$y = C_c \mathbf{x}_c + D_c u \tag{8b}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{8c}$$

$$C_c = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \text{ and} \qquad D_c = \begin{bmatrix} d_1 \end{bmatrix}, \tag{8d}$$

where the components a_i are defined by the original characteristic polynomial

$$P = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0. \tag{9}$$

With A_c defined, the form of the feedback state model with feedback row vector $\mathbf{K}_c$ is:

$$A'_c = A_c - B_c \mathbf{K}_c, \qquad B'_c = B_c, \tag{10a}$$

$$C'_c = C_c - D_c \mathbf{K}_c, \text{ and} \qquad D'_c = D_c. \tag{10b}$$

A'_c deserves further attention. The special canonical form of A_c and B_c makes the expression for A'_c simply

$$A'_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 1 \\ -(a_0 + K'_1) & -(a_1 + K'_2) & \cdots & -(a_{n-1} + K'_n) \end{bmatrix}, \tag{11}$$

where K_i' is the row vector of gains in the phase-variable canonical basis. The design characteristic polynomial coefficients δ_i must equal the characteristic polynomial coefficients

$$\delta_i = a_i + K_{i+1}', \tag{12}$$

which gives

$$K_i' = \delta_{i-1} - a_{i-1}. \tag{13}$$

This yields $\mathbf{K}'$. If we equate the feedback

$$\begin{aligned} \mathbf{K}\mathbf{x} &= \mathbf{K}'\mathbf{x_c} \quad \Longrightarrow \\ \mathbf{K} &= \mathbf{K}'T_c. \end{aligned} \tag{14}$$

Let $\mathcal{U}$ and $\mathcal{U}_c$ be the controllability matrices for the original basis and the phase-variable canonical basis, respectively. From Appendix B.02, we can compute the transformation matrix to be

$$T_c = \mathcal{U}_c \mathcal{U}^{-1}. \tag{15}$$

Steady-state error

We can use the gain N to drive the closed-loop steady-state error to zero for step inputs. The idea is that we can scale the input by the reciprocal of the closed-loop steady-state error. Let $G_{CL}(s)$ be the closed-loop transfer function. From the final value theorem for a unit step input,

$$N = \lim_{s \to 0,\, N \to 1} 1/G_{CL}(s). \tag{16}$$

If N is nonzero and finite, the response will have zero steady-state error. Although it is derived from unit step inputs, we can apply this formula to slowly varying inputs as well.

Example ss.sfdbck-1 re: state feedback pole placement design

Given the state-space model

$$A = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix},$$

design a controller with 15% overshoot and a settling time of 1 sec.

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