

B.02 Canonical forms of the state model

There are several canonical forms for the state equations, all of which can be found via basis transformations from other forms.

Phase-variable canonical form

The **phase-variable canonical form** is represented by the SISO¹ state model

$$\dot{\mathbf{x}}_c = \mathbf{A}_c \mathbf{x}_c + \mathbf{B}_c u \quad (1a)$$

$$y = \mathbf{C}_c \mathbf{x}_c + \mathbf{D}_c u \quad (1b)$$

where

$$\mathbf{A}_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad (1c)$$

$$\mathbf{C}_c = [c_1 \quad c_2 \quad \cdots \quad c_n], \quad \text{and} \quad \mathbf{D}_c = [d_1]. \quad (1d)$$

In order to transform a SISO system $\{A, B, C, D\}$ with state vector \mathbf{x} to phase-variable canonical form, we change bases via the substitution of $\mathbf{x} = \mathbf{T}_c \mathbf{x}_c$ into the original system, which gives

$$\mathbf{A}_c = \mathbf{T}_c^{-1} \mathbf{A} \mathbf{T}_c, \quad \mathbf{B}_c = \mathbf{T}_c^{-1} \mathbf{B}, \quad (2a)$$

$$\mathbf{C}_c = \mathbf{C} \mathbf{T}_c, \quad \text{and} \quad \mathbf{D}_c = \mathbf{D}. \quad (2b)$$

The special form of [Equation 1](#) yields the following characteristic polynomial:

$$s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \quad (3)$$

Recall that eigenvalues of a system are invariant to basis change, and therefore so is its characteristic polynomial. From this we can conclude that \mathbf{A}_c can be completely determined by finding the characteristic polynomial of the

phase-variable canonical form

1. There are phase-variable canonical forms for MIMO systems as well, but these are less standardized.

original matrix A . B_c is already fully determined, but C_c and D_c remain undetermined. They may be found by discovering the transformation matrix T_c and substituting it into [Equation 2](#).

Finding the phase-variable canonical transformation

The phase-variable canonical transformation matrix T_c can be found by relating the controllability matrices of the original form and the canonical form.

Theorem B.4: phase-variable canonical transformation

The transformation matrix from a system representation with controllability matrix \mathcal{U} to a phase-variable canonical transformation with controllability matrix \mathcal{U}_c is

$$T_c = \mathcal{U}_c \mathcal{U}^{-1}. \quad (4)$$

By the Definition of the controllability matrix, the original controllability matrix is

$$\mathcal{U} = [B \mid AB \mid A^2B \mid \dots \mid A^{n-1}B] \quad (5)$$

and that of the canonical form is

$$\mathcal{U}_c = [B_c \mid A_c B_c \mid A_c^2 B_c \mid \dots \mid A_c^{n-1} B_c]. \quad (6)$$

Note that \mathcal{U} and \mathcal{U}_c are both known from above. We relate the two forms by applying [Equation 2](#) to [Equation 6](#) to yield

$$\mathcal{U}_c = [T_c^{-1}B \mid T_c^{-1}AB \mid T_c^{-1}A^2B \mid \dots \mid T_c^{-1}A^{n-1}B] \quad (7a)$$

$$= T_c \mathcal{U}, \quad (7b)$$

to yield

$$T_c = \mathcal{U}_c \mathcal{U}^{-1}.$$

C

Physical topics