

## A.01 Complex functions

A **complex function**  $f$  maps a subset of the complex plane to the complex plane (i.e.  $f: \mathbb{C} \rightarrow \mathbb{C}$ ). For instance, a complex function  $f$  can map a single complex number  $s_0$  to another  $s_1 = f(s_0)$ .

A **curve** in the complex plane is defined as a continuous function mapping a closed interval of the reals to the complex plane. A **contour** is defined as a directed curve consisting of a finite set of directed smooth curves, the final endpoint of which is identical to the starting point (Fig. A.01.1 shows a plot of a contour  $\Gamma$ ).

A contour can be mapped by a complex function, and this is our primary concern. The image of a contour  $\Gamma$  mapped by a complex function  $f$  is itself a contour  $f(\Gamma)$ , as shown in Fig. A.01.2.

Complex functions are of interest in control theory because transfer functions, one of the central mathematical objects of control theory, are complex functions. The utility of evaluating and mapping contours with complex functions arises especially in root-locus design and frequency response design (especially for the Nyquist stability criterion).

### Example A.01-1

Map the complex point  $s = 1 + j3$  with the transfer (complex) function

$$H(s) = \frac{s + 4}{s - 1}$$

Sometimes we say that we are "evaluating" the

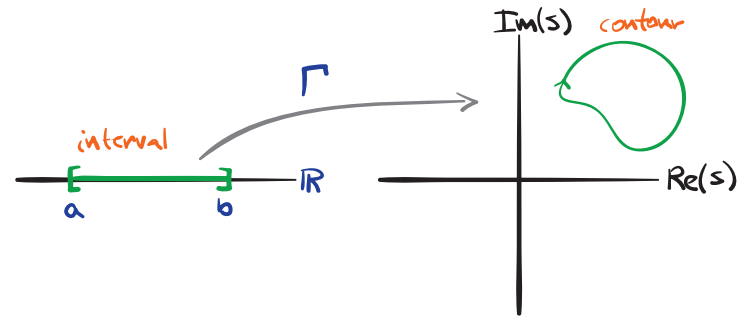


Figure A.01.1: illustrating the definition of a complex contour  $\Gamma$ .

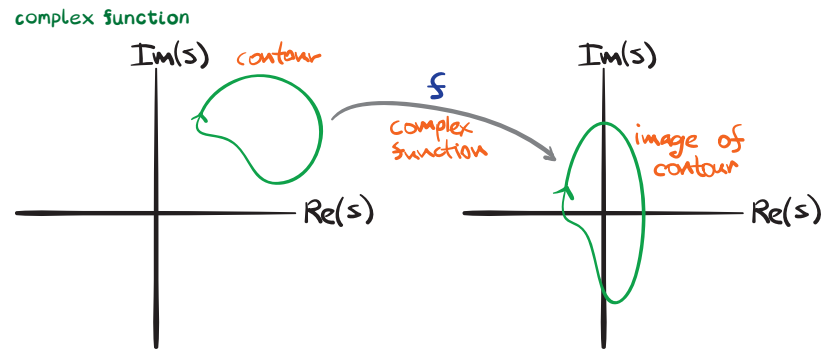


Figure A.01.2: a representation of a complex function  $f$  mapping a contour.

curve  
contour

re: transfer function mapping a single point

transfer function at the point  $s = 1 + j3$ .

### A geometric interpretation of complex functions

It is often helpful to interpret the complex mapping of a point or a contour geometrically. Let us consider a transfer (complex) function  $H(s)$  with complex zeros  $z_i$ , complex poles  $p_j$ , and real scaling factor  $k$ . Considering each factored term of the transfer function in terms of its magnitude and phase, we can write the magnitude and phase of the transfer function as follows.

**Equation 1 magnitude and phase of a transfer function**

We can interpret this geometrically as follows. Let us consider the evaluation of Eq. 1 at a specific complex value  $\psi$ . The differences  $\psi - z_i$  and  $\psi - p_i$  can be thought of as vectors in the complex plane with tails at  $z_i$  and  $p_i$  and heads at  $\psi$ . Fig. A.01.3 shows this geometric interpretation with  $p_{1,2} = -3 \pm j3$ ,  $z_1 = 1$ , and  $\psi = 3 + j4$ .

#### Example A.01-2

Let  $I = [0, 2\pi]$ . Let the contour  $\Gamma : I \rightarrow \mathbb{C}$  be defined parametrically, with  $t \in I$ , as

$$\Gamma(t) = \sin t + j \cos t.$$

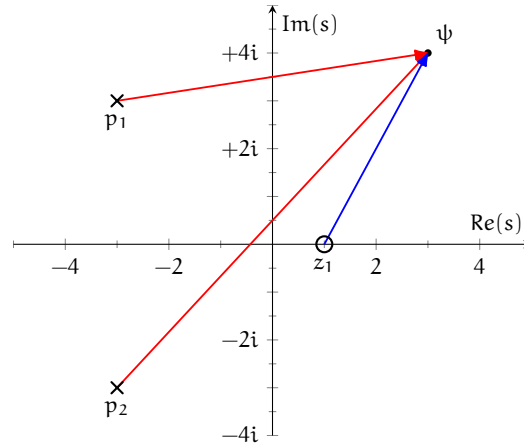


Figure A.01.3: an example of a geometric interpretation of the evaluation of a complex function with poles  $p_{1,2}$  and zero  $z_1$  at a complex value  $s = \psi$ .

#### re: transfer function mapping a contour

```

1 \[CapitalGamma][t_] := {Sin[t], Cos[t]}
2 H[s_] := (s + 1)/(s^2 + 2*s + 2);
3
4 ps = {Blue, Arrowheads[{{0, .05, .05, .05}}]};
5 mappingcontour = Animate[
6   {
7     ParametricPlot[
8       \[CapitalGamma][t], {t, 0, T},
9       PlotRange -> {-1, 1},
10      PlotStyle -> ps,
11      PlotLabel -> "\[CapitalGamma]"
12    ] /.
13     Line -> Arrow,
14    ParametricPlot[
15      H[Complex @@ \[CapitalGamma][t]] // {Re[#], Im[#]} &,
16      {t, 0.001, T},
17      PlotRange -> {-1.5, 1.5},
18      PlotStyle -> ps,
19      PlotLabel -> "H(\[CapitalGamma])"
20    ] /.
21     Line -> Arrow
22  } // GraphicsRow,
23  {T, 0, 2*\[Pi]}
24 ]

```

Figure A.01.5: a basic Mathematica script for visualizing the transfer function mapping of Example A.01-2. A more thorough notebook is available [here](#).

• Map  $\Gamma$  with the transfer function

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

and plot the result.

B

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## Linear systems theory topics