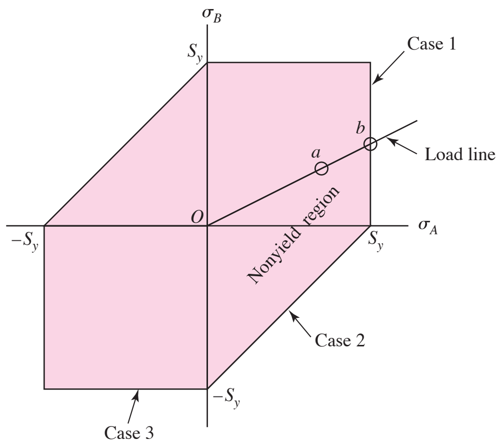


Figure 5–7

The maximum-shear-stress (MSS) theory yield envelope for plane stress, where σ_A and σ_B are the two nonzero principal stresses.

of a member. If the load is increased, it is typical to assume that the principal stresses will increase proportionally along the line from the origin through point a. Such a *load line* is shown. If the stress situation increases along the load line until it crosses the stress failure envelope, such as at point b, the MSS theory predicts that the stress element will yield. The factor of safety guarding against yield at point a is given by the ratio of strength (distance to failure at point b) to stress (distance to stress at point a), that is $n = Ob/Oa$.

Note that the first part of Eq. (5–3), $\tau_{max} = S_y/2n$, is sufficient for design purposes provided the designer is careful in determining τ_{max}. For plane stress, Eq. (3–14) *does not always predict* τ_{max}. However, consider the special case when one normal stress is zero in the plane, say σ_x and τ_{xy} have values and $\sigma_y = 0$. It can be easily shown that this is a Case 2 problem, and the shear stress determined by Eq. (3–14) *is* τ_{max}. Shaft design problems typically fall into this category where a normal stress exists from bending and/or axial loading, and a shear stress arises from torsion.

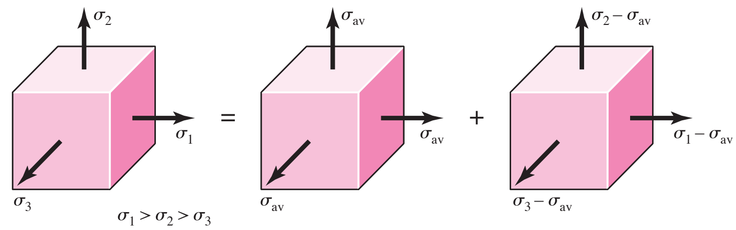
5–5 Distortion-Energy Theory for Ductile Materials

The *distortion-energy theory* predicts that *yielding occurs when the distortion strain energy per unit volume reaches or exceeds the distortion strain energy per unit volume for yield in simple tension or compression of the same material*.

The distortion-energy (DE) theory originated from the observation that ductile materials stressed hydrostatically (equal principal stresses) exhibited yield strengths greatly in excess of the values given by the simple tension test. Therefore it was postulated that yielding was not a simple tensile or compressive phenomenon at all, but, rather, that it was related somehow to the angular distortion of the stressed element. To develop the theory, note, in Fig. 5–8a, the unit volume subjected to any three-dimensional stress state designated by the stresses σ_1, σ_2, and σ_3. The stress state shown in Fig. 5–8b is one of hydrostatic normal stresses due to the stresses σ_{av} acting in each of the same principal directions as in Fig. 5–8a. The formula for σ_{av} is simply

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \tag{a}$$

Thus the element in Fig. 5–8b undergoes pure volume change, that is, no angular distortion. If we regard σ_{av} as a component of σ_1, σ_2, and σ_3, then this component can be subtracted from them, resulting in the stress state shown in Fig. 5–8c. This element is subjected to pure angular distortion, that is, no volume change.



(*a*) Triaxial stresses (*b*) Hydrostatic component (*c*) Distortional component

Figure 5–8

(*a*) Element with triaxial stresses; this element undergoes both volume change and angular distortion. (*b*) Element under hydrostatic normal stresses undergoes only volume change. (*c*) Element has angular distortion without volume change.

The strain energy per unit volume for simple tension is $u = \frac{1}{2}\epsilon\sigma$. For the element of Fig. 5–8*a* the strain energy per unit volume is $u = \frac{1}{2}[\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3]$. Substituting Eq. (3–19) for the principal strains gives

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \tag{b}$$

The strain energy for producing only volume change u_v can be obtained by substituting σ_{av} for σ_1, σ_2, and σ_3 in Eq. (*b*). The result is

$$u_v = \frac{3\sigma_{av}^2}{2E}(1 - 2\nu) \tag{c}$$

If we now substitute the square of Eq. (*a*) in Eq. (*c*) and simplify the expression, we get

$$u_v = \frac{1 - 2\nu}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1\right) \tag{5–7}$$

Then the distortion energy is obtained by subtracting Eq. (5–7) from Eq. (*b*). This gives

$$u_d = u - u_v = \frac{1 + \nu}{3E}\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right] \tag{5–8}$$

Note that the distortion energy is zero if $\sigma_1 = \sigma_2 = \sigma_3$.

For the simple tensile test, at yield, $\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$, and from Eq. (5–8) the distortion energy is

$$u_d = \frac{1 + \nu}{3E}S_y^2 \tag{5–9}$$

So for the general state of stress given by Eq. (5–8), yield is predicted if Eq. (5–8) equals or exceeds Eq. (5–9). This gives

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \geq S_y \tag{5–10}$$

If we had a simple case of tension σ, then yield would occur when $\sigma \geq S_y$. Thus, the left of Eq. (5–10) can be thought of as a *single, equivalent,* or *effective stress* for the entire general state of stress given by σ_1, σ_2, and σ_3. This effective stress is usually

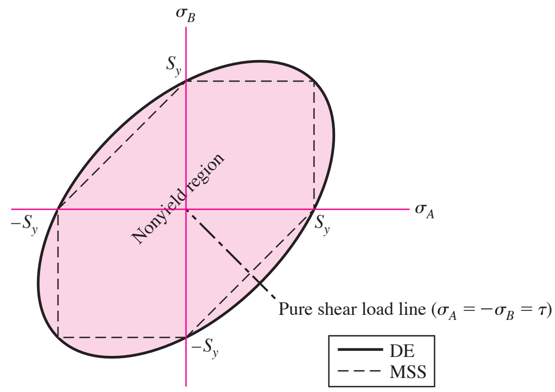


Figure 5–9

The distortion-energy (DE) theory yield envelope for plane stress states. This is a plot of points obtained from Eq. (5–13) with $\sigma' = S_y$.

called the *von Mises stress,* σ', named after Dr. R. von Mises, who contributed to the theory. Thus Eq. (5–10), for yield, can be written as

$$\sigma' \geq S_y \tag{5–11}$$

where the von Mises stress is

$$\sigma' = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \tag{5–12}$$

For plane stress, the von Mises stress can be represented by the principal stresses σ_A, σ_B, and zero. Then from Eq. (5–12), we get

$$\sigma' = \left(\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2\right)^{1/2} \tag{5–13}$$

Equation (5–13) is a rotated ellipse in the σ_A, σ_B plane, as shown in Fig. 5–9 with $\sigma' = S_y$. The dotted lines in the figure represent the MSS theory, which can be seen to be more restrictive, hence, more conservative.[4]

Using *xyz* components of three-dimensional stress, the von Mises stress can be written as

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)\right]^{1/2} \tag{5–14}$$

and for plane stress,

$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{1/2} \tag{5–15}$$

The distortion-energy theory is also called:

- The von Mises or von Mises–Hencky theory
- The shear-energy theory
- The octahedral-shear-stress theory

Understanding octahedral shear stress will shed some light on why the MSS is conservative. Consider an isolated element in which the normal stresses on each surface are

[4]The three-dimensional equations for DE and MSS can be plotted relative to three-dimensional σ_1, σ_2, σ_3, coordinate axes. The failure surface for DE is a circular cylinder with an axis inclined at 45° from each principal stress axis, whereas the surface for MSS is a hexagon inscribed within the cylinder. See Arthur P. Boresi and Richard J. Schmidt, *Advanced Mechanics of Materials,* 6th ed., John Wiley & Sons, New York, 2003, Sec. 4.4.