Linear independence + bases

(Based on Bullo + Lewis's Geometric Control of Mechanical Systems, §22)

For the following definitions, let V be a real vector space and $c_1, c_2, \dots, c_k \in \mathbb{R}$.

Definition A set $S \subset V$ of vectors is *linearly independent* if, for every finite subset $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\} \subset S$ the equality

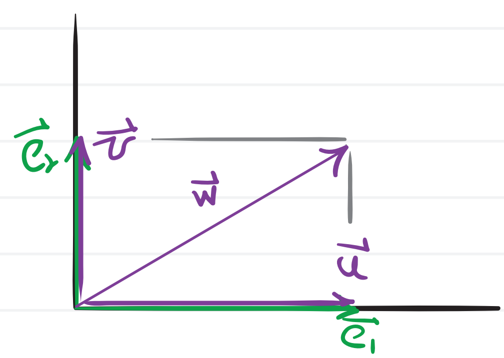
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \vec{0}$$

implies that $c_1 = c_2 = \dots = c_k = 0$.

Essentially, vectors are linearly independent if they can't be scaled and added together to equal each other.

Example Let $\vec{u} = 1\vec{e}^{\,1} + 0\vec{e}^{\,2}$, $\vec{v} = 0\vec{e}^{\,1} + 1\vec{e}^{\,2}$, and $\vec{w} = 1\vec{e}^{\,1} + 1\vec{e}^{\,2}$. Are $\vec{u}$ and $\vec{v}$ linearly independent? $\vec{u}$ and $\vec{w}$? $\vec{u}, \vec{v}$, and $\vec{w}$?

We can't scale $\vec{u}$ to equal $\vec{v}$. There's no $\vec{e}^{\,2}$-component in $\vec{u}$ and no $\vec{e}^{\,1}$-component in $\vec{v}$. Therefore, $\vec{u}$ + $\vec{v}$ are linearly *independent*.



Since $\vec{w}$ has an $\vec{e}^{\,1}$-component and $\vec{v}$ has none, $\vec{w}$ + $\vec{v}$ are *independent*.

If we add $\vec{u}$ + $\vec{v}$, we get $\vec{w}$ ($\vec{u} + \vec{v} = \vec{w}$). Therefore, $\vec{u}, \vec{v}$, + $\vec{w}$ are linearly *dependent*.

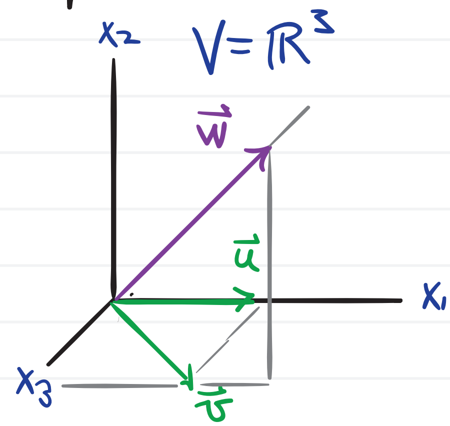
Definition A set of vectors $S \subset V$ **generates** a vector space V if every vector $\vec{u} \in V$, can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_k \in S$. That is, for some choice of constants $c_1, c_2, \cdots, c_k \in \mathbb{R}$,

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k .$$

if $S \subset V$ generates V, we can write $V = \mathrm{span}(S)$.

If S generates V, we say that S spans V or V is spanned by S.

For instance, if V is $\mathbb{R}^3$, it isn't spanned by $\vec{u}$ and $\vec{v}$. But $\vec{u}$ and $\vec{v}$ do span $\mathbb{R}^2$.



Definition A **basis** for a vector space V is a collection of vectors that is linearly independent and that generates V.

We typically denote a basis b of $\mathbb{R}^n$ by $b = (\vec{b}^1, \vec{b}^2, \cdots, \vec{b}^n)$.

Essentially, a basis is a minimal set of vectors that can be scaled and added together to form every other vector in the vector space.

The scaling factors are called **components** of the vectors. Sometimes we write vectors only with components (and without basis vectors), like

$$\vec{u} = [1.2 \quad 6.7 \quad -3.0] \quad \text{or} \quad \vec{v} = \begin{bmatrix} 0.2 \\ 7.6 \\ -2.7 \end{bmatrix}$$

This notation always implies some basis.

Definition The standard basis for $\mathbb{R}^n$ is given by

$$\vec{e}^1 = (1, 0, \cdots, 0), \quad \vec{e}^2(0, 1, \cdots, 0), \quad \cdots, \quad \vec{e}^n = (0, 0, \cdots, 1) .$$

Example Write the vector shown in the figure in the standard basis.

$$\vec{v} = 3\vec{e}^1 + 2.5\vec{e}^2 + 1\vec{e}^3 .$$

$$= \begin{bmatrix} 3 \\ 2.5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2.5 & 1 \end{bmatrix}$$