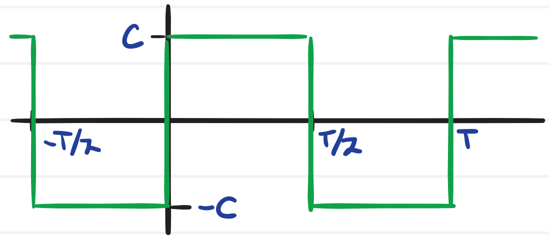
Fourier series example

Consider the square wave function



$$y(t) = \begin{cases} C & \text{for } 0 \leq \mathrm{mod}(t,T) < 1/2 \\ -C & \text{for } T/2 \leq \mathrm{mod}(t,T) < T. \end{cases}$$

We will perform a Fourier series analysis of this function and synthesize the first number of harmonics to see how well they approximate $y(t)$.

Fourier analysis

We apply the equations of Fourier analysis:

$$A_0 \equiv \frac{2}{T}\int_{-T/2}^{T/2} y(t)\, dt$$

$$A_n \equiv \frac{2}{T}\int_{-T/2}^{T/2} y(t) \cos\frac{2\pi n t}{T}\, dt$$

$$B_n \equiv \frac{2}{T}\int_{-T/2}^{T/2} y(t) \sin\frac{2\pi n t}{T}\, dt$$

Aside

Fourier synthesis

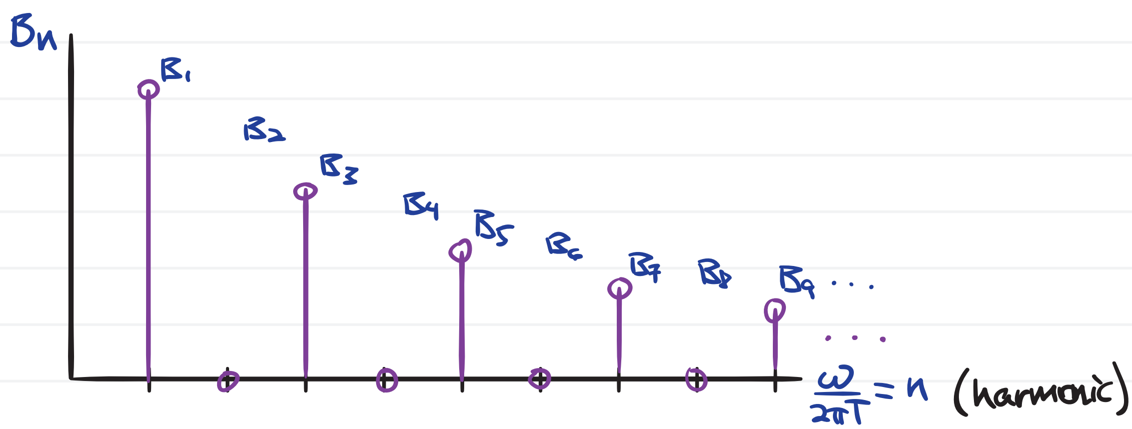
We can now synthesize the first p terms of the series. The full series is

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2\pi n t}{T} + B_n \sin\frac{2\pi n t}{T}\right)$$

We can say that, for sufficiently large p,

Let's compute the first few harmonic amplitudes:

We can visualize these harmonic contributions on a frequency spectrum as follows.

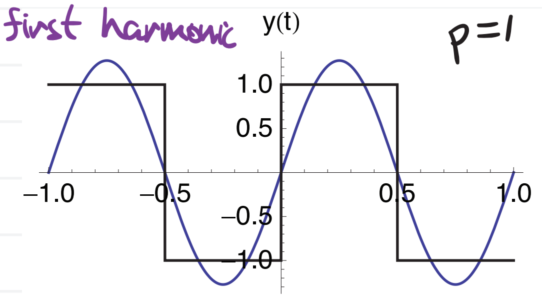


Thinking about a signal $y(t)$ in terms of its harmonics and its frequency spectrum is common practice.

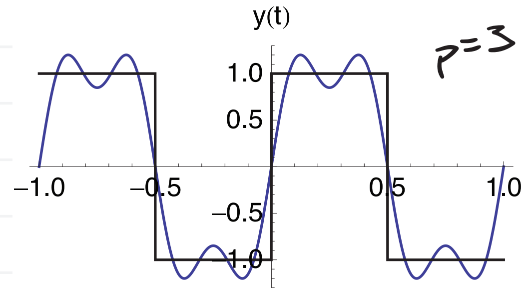
Now we can try to visualize the summation of the harmonics in the Fourier series.

first harmonic

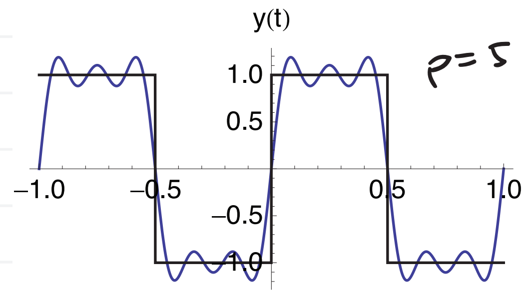
$p=1$



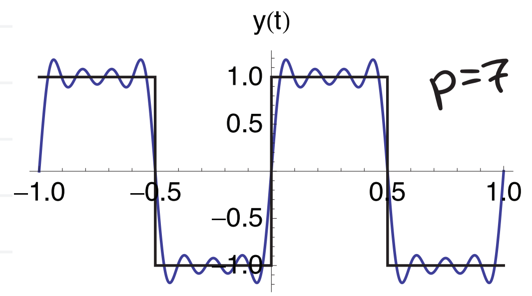
$p=3$



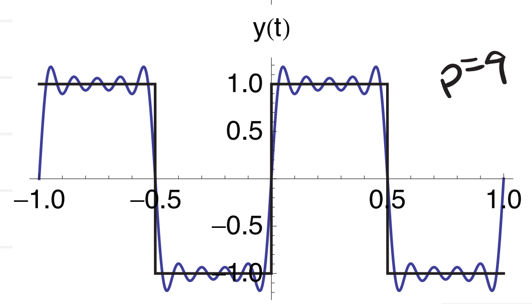
$p=5$



$p=7$

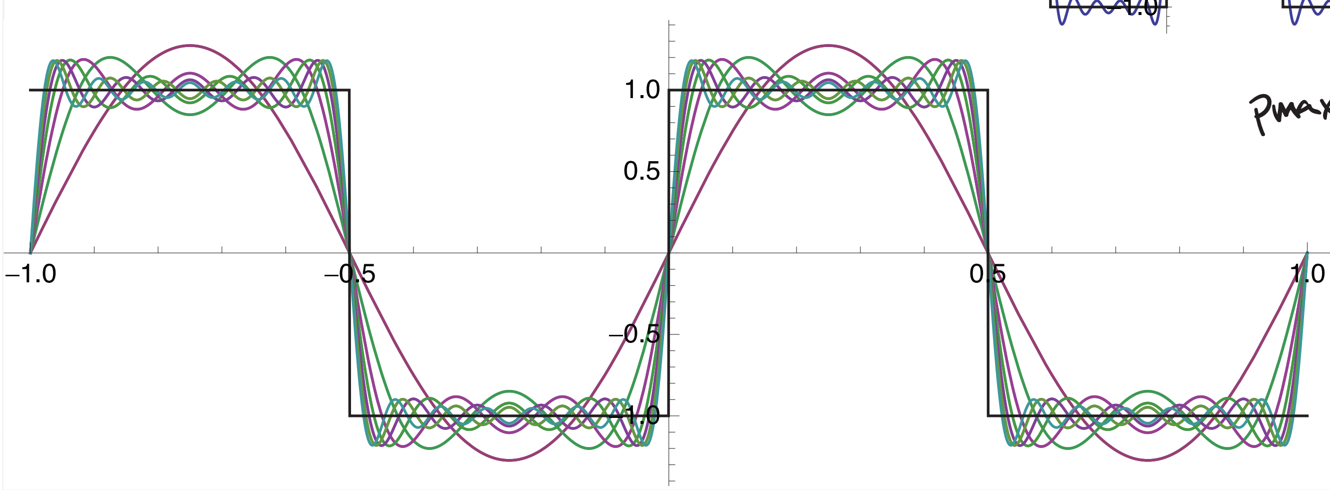


$p=9$



Below, we can begin to see that as p increases, the series doesn't converge near the discontinuities. This is called the Gibbs phenomenon.

The most important takeaway is that periodic functions may be represented as sums of sines and cosines.



$p_{max}=13$