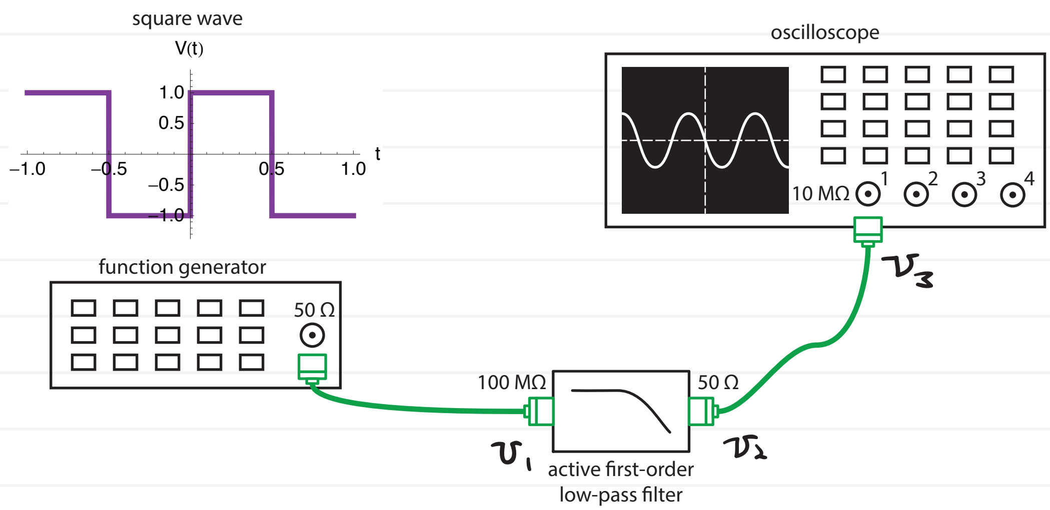
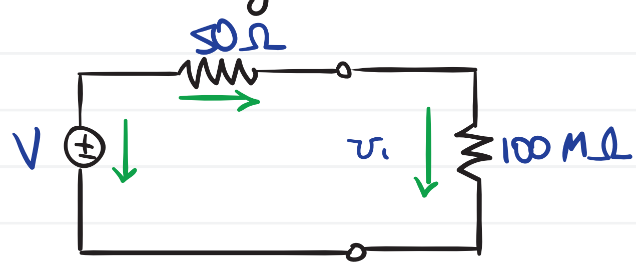
Frequency response example

Consider the measurement setup below. A square wave $V(t)$ is supplied by the function generator at 1 Hz. This signal is connected to the input of an active low-pass filter with input impedance $100\ \mathrm{M\Omega}$, cutoff angular frequency $\omega_c = 2\pi$ rad/s, and output impedance $50\ \Omega$. What will the signal look like on the oscilloscope?

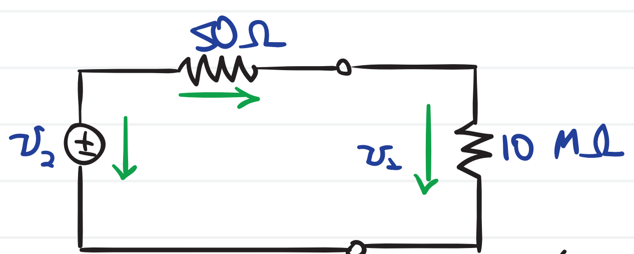


Solution

1. Circuit analysis.



$$\xrightarrow{\text{voltage divider}} \quad v_1 = \frac{100 \cdot 10^6}{100 \cdot 10^6 + 50} V \cong V$$



$$\xrightarrow{\text{voltage divider}} \quad v_3 = \frac{10 \cdot 10^6}{10 \cdot 10^6 + 50} v_2 \cong v_2$$

(This is why we like instruments to have low output impedance and high input impedance.)

2. Fourier series of input to the filter.
We already did this. Since the function is odd, $A_0 = A_n = 0$.
Recall that the result was

$$v_i(t) = \sum_{n=1}^{\infty} B_n \sin \frac{2\pi n t}{T} \quad \text{with}$$

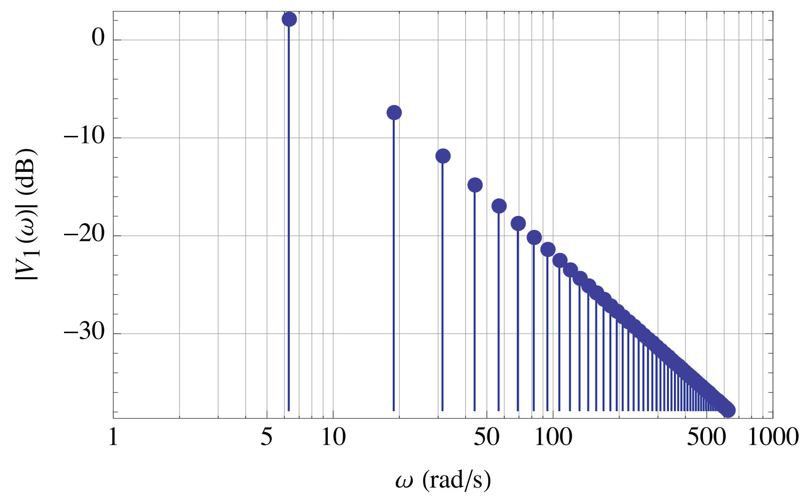
$$B_n = \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{4}{n\pi} & \text{for } n \text{ odd} \end{cases} .$$

The frequency "line" spectrum looks like the following.

The magnitudes are computed by

$$|V_1(\omega)| = \sqrt{A_n^2 + B_n^2}$$

with $\omega = \frac{2\pi n}{T}$.



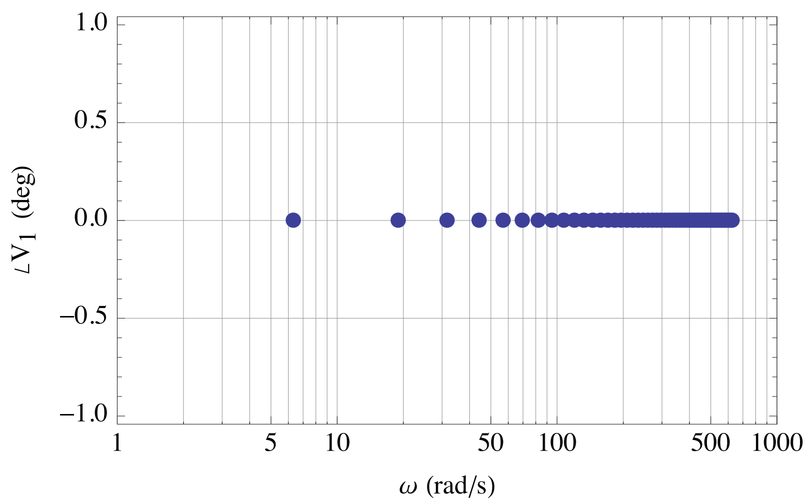
The phases are computed by

$$\angle V_1(\omega) = \operatorname{atan} \frac{A_n}{B_n}$$

$$= \operatorname{atan} \frac{0}{B_n}$$

$$= 0$$

with $\omega = \frac{2\pi n}{T}$.



3. Frequency response function of the filter.

As we covered previously, a first-order low-pass frequency filter has the input-output ODE

$$\tau \frac{dv_2}{dt} + v_2 = v_1 \quad \text{where} \quad \tau \equiv \frac{1}{\omega_c}$$

cutoff angular freq.

Fourier transform and solve for the freq. response func. $H(\omega)$.

$$\mathcal{F}\left\{\tau \frac{dv_2}{dt}\right\} + \mathcal{F}\{v_2\} = \mathcal{F}\{v_1\} \quad \Longrightarrow$$

$$\tau (j\omega) V_2(\omega) + V_2(\omega) = V_1(\omega)$$

$$H(\omega) \equiv \frac{V_2(\omega)}{V_1(\omega)} = \frac{1}{1 + j\tau\omega}.$$

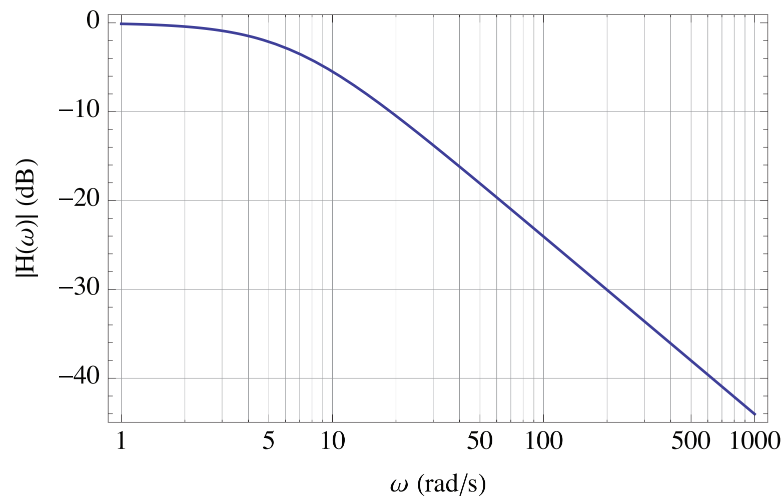
4. Magnitude + phase of $H(\omega)$.

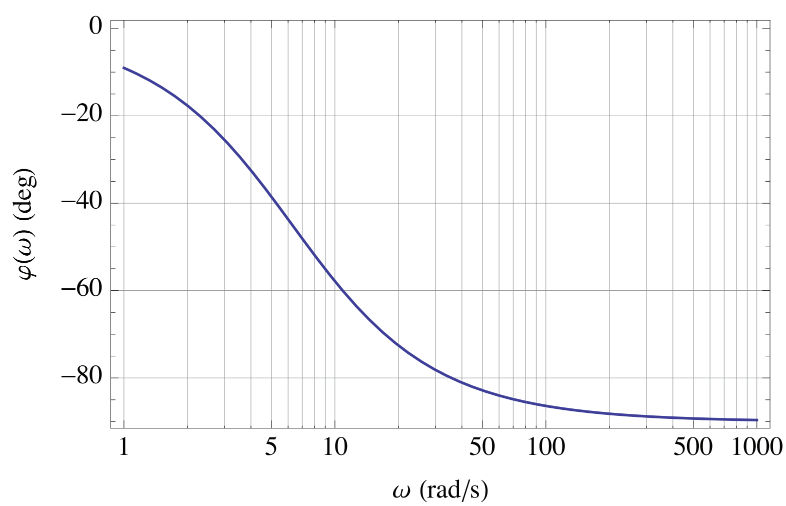
$$|H(\omega)| = \frac{1}{\sqrt{1+\tau^2\omega^2}} = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^2}}$$

$$\phi(\omega) = \arctan \frac{-\frac{\tau\omega}{1+\tau^2\omega^2}}{\frac{1}{1+\tau^2\omega^2}} = -\arctan\left(\frac{\omega}{\omega_c}\right)$$

$$\left(H(\omega) = \frac{1}{1+\tau^2\omega^2} - j\frac{\tau\omega}{1+\tau^2\omega^2}\right)$$

5. Bode plot.





6. The principle of superposition for these SISO systems allows us to break down this problem into parts. The input is a sum of sinusoids with various amplitudes B_n. We will find the output for each and sum them.

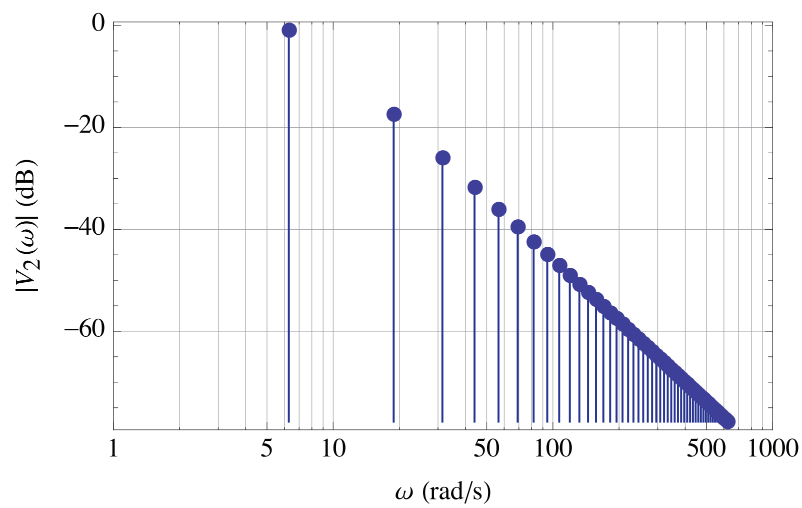
Recall that the steady-state response to a sinusoidal input is

$$v_{2n}(t) = B_n |H(\omega_n)| \sin(\omega_n t + \cancel{\psi_n}^{\,0 \text{ (note: if } A_n \neq 0,\ \psi_n \neq 0)} + \phi(\omega_n)).$$

From superposition, then,

$$v_2(t) = \sum_{n=1}^{\infty} v_{2n}(t) = \sum_{n=1}^{\infty} B_n |H(\omega_n)| \sin(\omega_n t + \phi(\omega_n)).$$

7. So we have a new frequency spectrum.



8. From our circuit analysis,

$$v_3(t) \hat{=} v_2(t).$$