Homogeneous state response

Recall that we can write the state + output equations for an LTI system as

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$
$$\vec{y} = C\vec{x} + D\vec{u} \quad .$$

We can solve these equations both analytically (SD Ch 10) + numerically (SD Ch 11).

Recall that when solving for scalar, first-order, homogeneous solutions we found them to be of the form

$$\dot{x} = a x \quad \Longrightarrow \quad x_h(t) = e^{at}\, x(0),$$

where $a \in \mathbb{R}$ is the system parameter and $x(0)$ the i.c.

The exponential term has the power series representation

$$e^{at} = 1 + at + \frac{a^2}{2!}t^2 + \frac{a^3}{3!}t^3 + \cdots + \frac{a^k}{k!}t^k \cdots \quad .$$

Now, consider the homogeneous vector state equation $\dot{\vec{x}} = A\vec{x}(0)$. Let's sneakily guess that is has solution

$$\dot{\vec{x}} = A\vec{x} \quad \Longrightarrow \quad \vec{x}_h(t) = \left(I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^k}{k!}t^k + \cdots\right)\vec{x}(0) \quad .$$

Let's check this guess. The time-derivative of $x_h(t)$ is

$$\dot{\vec{x}}_h = \left(0 + A + A^2 t + A^3\frac{t^2}{2!} + \cdots\right)\vec{x}(0)$$

$$= A\left(I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^k}{k!}t^k + \cdots\right)\vec{x}(0)$$

$$= A\vec{x}_h \quad ,$$ which is the homogeneous state equation!

The magical matrix series is called the matrix exponential, and denoted

$$\boxed{e^{At} \equiv \left(I + At + \frac{A^2}{2!}t^2 + \cdots + \frac{A^k}{k!}t^k + \cdots\right)}.$$

Peano-Baker series

The state transition matrix $\Phi(t)$ is defined as

$$\boxed{\Phi(t) \equiv e^{At}}.$$

However, $\Phi(t)$ is usually computed from a method other than from its definition as the matrix exponential. We will cover this method later.

Example

Example 10.1 of SD gives that, for $A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}$,

$\Phi(t) = \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix}$. So, for some i.c.s $\vec{x}(0)$, the homogeneous state response is

$$\begin{aligned}
\vec{x}_h(t) &= \Phi(t)\,\vec{x}(0) \\
&= \begin{bmatrix} e^{-2t} & 0 \\ e^{-t} - e^{-2t} & e^{-t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} x_1(0) \\ (e^{-t} - e^{-2t}) x_1(0) + e^{-t} x_2(0) \end{bmatrix}
\end{aligned}$$

Example 10.1 requires that power series are recognized as describing exponential functions (a difficult task).

Let's use Matlab to compute the first few terms of the matrix exponential e^{At} and see if we can get it to converge to the analytic result.