State equation diagonalization

It is useful to transform a system's state vector $\vec{x}$ into a special basis that "diagonalizes" the A-matrix. For systems with n distinct eigenvalues, this is always possible. In the diagonalized form, it will be relatively easy to solve for the state transition matrix.

Changing bases in the state equation

As with all basis transformations, it can be written in the form

where P is the transformation matrix, $\vec{x}$ is the original state vector, and $\vec{x}'$ is the new state vector.

Substituting into the state + output equations, the state and output equations in the new basis are

$$\boxed{}\ .$$

Modal + eigenvalue matrices

Definition — Let a state equation have matrix A with n distinct eigenvalues (λ_i) and eigenvectors $(\vec{m}_i)$. Let Λ be defined as

$$\boxed{\Lambda \equiv \left[\right]}$$

called the eigenvalue matrix.

Furthermore, let M be defined as $\boxed{}$, called the modal matrix.

Diagonalization of the state equation

Let $\vec{x}' = M^{-1}\vec{x}$ (as long as there are n distinct eigenvalues, M is invertible), where M is the modal matrix for the state equation in $\vec{x}$.

Then $\dot{\vec{x}}' = (\underline{M^{-1}AM})\vec{x}' + (M^{-1}B)\vec{u}$ from the eigen. prob.

So $\boxed{\qquad\qquad\qquad\qquad}$.

Recall that Λ is completely diagonal.

An easier way to compute the state transition matrix

Recall our definition of the state transition matrix $\Phi(t) \equiv e^{At}$. Directly applying this to the diagonalized system,

$$\boxed{\Phi'(t) = e^{\Lambda t} = \begin{bmatrix} \quad \end{bmatrix}} .$$

So the "homogeneous" (unforced) solution is

$$\boxed{\vec{x}_h'(t) = \Phi'(t)\vec{x}'(0) = x_1'(0)e^{\lambda_1 t} + x_2'(0)e^{\lambda_2 t} + \dots + x_n'(0)e^{\lambda_n t}} ,$$

where $\vec{x}'(0) = M^{-1}\vec{x}(0) \implies \vec{x}(0) = M\vec{x}'(0)$.

Recall that $\vec{x}_h(t) = \Phi(t)\vec{x}(0)$. Then

$$\left\| \ \boxed{\qquad\qquad} . \ \right\|$$