Fundamentals of frequency response

Let's restrict ourselves to systems with sinusoidal inputs

Using Euler's formula, we can write a general complex sinusoid in exponential form:

$$u(t) = U(s)\, e^{st}\Big|_{s=j\omega} = U(j\omega)\, e^{j\omega t} = U(j\omega)(\cos \omega t + j \sin \omega t).$$

We know we can find a transfer function $H(s)$ for a system with this input because it is exponential. We are usually concerned about the steady-state response of systems to sinusoidal inputs. These occur after the transient response has decayed; that is, when the response is just the particular solution.:

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = y_p(t).$$

We can write this in terms of a transfer function:

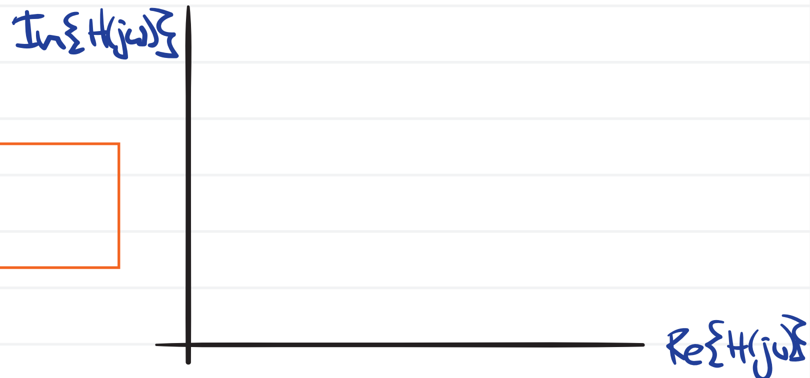
So the output complex amplitude is $H(j\omega)$ times the input amplitude. I.e.

$$\boxed{}\,.$$

$H(j\omega)$ is called the frequency response function. We see that it can be directly derived from the transfer function with $s \mapsto j\omega$.

$H(j\omega)$ is a complex function, with a real $\mathrm{Re}\{H(j\omega)\}$ and an imaginary $\mathrm{Im}\{H(j\omega)\}$ part. For a given input frequency ω, $H(j\omega)$ is a complex number, which is easily represented in the complex plane

The polar representation in terms of a magnitude



and a phase

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Recall that we have been considering a complex sinusoidal input. However, we want to know the steady-state response to a real sinusoidal input. We can write this as a superposition of complex sinusoids:

$$A\sin(\omega t) = \frac{A}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$A\cos(\omega t) = \frac{A}{2}\left(e^{j\omega t} + e^{-j\omega t}\right).$$

Using the principle of superposition, it can be shown that for $u(t) = A\sin(\omega t + \psi)$ is

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This is a very important result. See SD Figure 14.3.