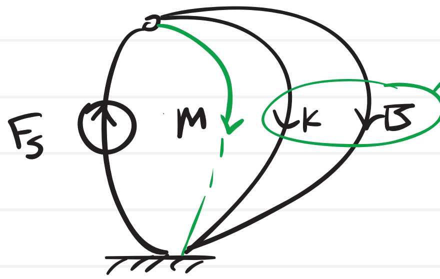
14.14: Vibration example

$$\frac{\frac{1}{Z_{KB}}}{\frac{1}{Z_{KB}}+\frac{1}{Z_m}} = \frac{Z_m}{Z_m+Z_{KB}}$$

Linear graph

primary: f_{KB}, v_m
secondary: v_{KB}, f_m



$$Z_{KB} = \frac{Z_K Z_B}{Z_K+Z_B} = \frac{(s/K)(1/B)}{s/K+\frac{1}{B}} = \frac{s}{Bs+K}$$

Variables + system order

State variables: v_m, f_K $\Longrightarrow$ $\vec{x} = \begin{bmatrix} v_m \\ f_K \end{bmatrix}$

System order: $n=2$

Elemental equations

m	$\dot{v}_m = \frac{1}{m} f_m$
K	$\dot{f}_K = K v_K$
B	$f_B = B v_B$

$\Longrightarrow$

$$v_m = \underbrace{\frac{1}{ms}}_{Z_m} f_m = Z_m f_m$$

$$\left.\begin{aligned} v_K &= Z_K f_K \\ v_B &= Z_B f_B \end{aligned}\right\} v_{KB} = Z_{KB} f_{KB}$$

Continuity

$\longrightarrow$ switching gears: impedances

Use the general method. Continuing with cont:

$$F_S = f_m + f_{KB}$$

Compatibility

$$v_m = v_{KB}$$

Algebra

We want the transfer function $\frac{f_{KB}}{F_S}$. We know that we can solve for $f_{KB}(F_S)$ from the continuity, compatibility, and elemental equations.

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & -Z_m & 0 & 0 \\ 0 & 0 & 1 & -Z_{KB} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} v_m \\ f_m \\ v_{KB} \\ f_{KB} \end{bmatrix}}_{\vec{v}_1} = \underbrace{\begin{bmatrix} 0 \\ F_S \\ 0 \\ 0 \end{bmatrix}}_{\vec{v}_2}$$

$$\vec{v}_1 = M^{-1}\vec{v}_2$$

Therefore,

$$f_{KB} = \frac{Z_m}{Z_m + Z_{KB}} F_s .$$

$$\frac{f_{KB}}{F_s} = \frac{1/ms}{1/ms + \frac{s}{Bs+k}} = \frac{Bs+k}{Bs+k+ms^2} = \frac{Bs+K}{ms^2+Bs+K}$$

$$= \frac{\frac{B}{m}s + \frac{k}{m}}{s^2 + \frac{B}{m}s + \frac{k}{m}}$$

Frequency response function

$$H(j\omega) = \left.\frac{f_{KB}(s)}{F_s(s)}\right|_{s=j\omega} = \frac{(k) + j(B\omega)}{(-m\omega^2 + k) + j(B\omega)}$$

Rationalize it:

$$H(j\omega) = \frac{(-m\omega^2+k) - j(B\omega)}{(-m\omega^2+k) - j(B\omega)} \cdot \frac{(k) + j(B\omega)}{(-m\omega^2+k) + j(B\omega)}$$

$$= \frac{(k)(-m\omega^2+k) + (B\omega)^2 + j\left(B\omega(-m\omega^2+k) - Bk\omega\right)}{(-m\omega^2+k)^2 + (B\omega)^2}$$

Question: what is the natural frequency? Damping ratio?

The standard form of the denominator of a second-order transfer function is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \Rightarrow \omega_n = \sqrt{\frac{K}{m}}$$

$$2\zeta\omega_n = \frac{B}{m} \Rightarrow \zeta = \frac{B}{2m\omega_n} = \frac{B}{2m\sqrt{\frac{K}{m}}} = \frac{B}{2\sqrt{Km}}$$

Bode plot

poles of t.f.: $p^2 + 2\zeta\omega_n p + \omega_n^2 = 0 \Rightarrow P_{1,2} = -\zeta\omega_n \pm \frac{1}{2}\sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}$

$\Rightarrow -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$ and, with a slight abuse of notation,

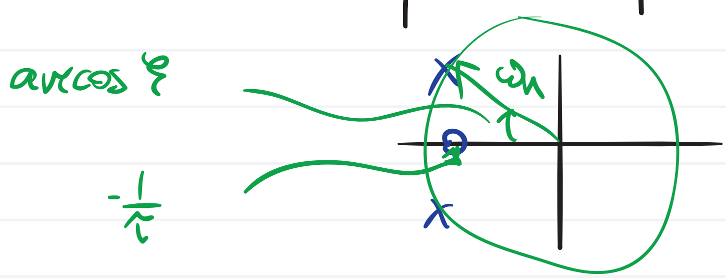
$$p_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

zeros of t.f.: $\frac{B}{m}z + \frac{k}{m} = 0 \Rightarrow z = -\frac{k}{B}$

$$\tau z + 1 = 0 \qquad \left(\tau = \frac{B}{k}\right)$$

$$\hookrightarrow z = \frac{-1}{\tau}$$

pole-zero plot



$m = 1000\ \text{kg} \qquad \omega_d = 20\ \text{rad/s} \qquad \xi = 0.2$

$$\omega_d = \omega_n\sqrt{1-\xi^2} \Rightarrow \omega_n = \frac{20}{\sqrt{1-0.2^2}} = 20.4\ \text{rad/s}$$

$$\omega_n = \sqrt{\frac{k}{m}} \Longrightarrow k = m\omega_n^2 = (1000)(20.4)^2 = 417\ \frac{\text{kN}}{\text{m}}$$

$$\tau = \frac{B}{k} = \frac{2\xi\omega_n m}{k} = \frac{2\xi\omega_n}{\omega_n^2} = \frac{2\xi}{\omega_n} = 0.0196\ \text{sec}$$

Let's write the transfer function as the product of three transfer functions:

$$(\qquad)(\tau s+1)\left(\frac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2}\right)$$

$$\frac{\frac{B}{m}s+\frac{k}{m}}{s^2+\frac{B}{m}s+\frac{k}{m}} = \frac{\frac{B}{m}s+\frac{k}{m}}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{k}{m}\cdot\frac{\frac{B}{k}s+1}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{k}{m}\,\frac{\tau s+1}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$= \frac{k}{m}\cdot\frac{1}{\omega_n^2}\cdot\frac{(\tau s+1)\cdot\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \underbrace{\left(\frac{k}{m}\cdot\frac{1}{\omega_n^2}\right)}_{\frac{k}{m}\frac{1}{k/m}=1}\cdot\underbrace{(\tau s+1)}\cdot\underbrace{\left(\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\right)}$$

Break frequencies: $1/\tau = 51$ $\omega_n = 20.4$