Fourier transforms + frequency response functions

Fourier transforms can be used to characterize the steady state response of a system to a sinusoidal input. Let $u(t)$ be the sinusoidal input, $y(t)$ be the output, and $a_k, b_\ell \in \mathbb{R}$. A SISO system has the representation

$$\boxed{\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \cdots + b_1 \frac{du}{dt} + b_0 u}.$$

We can take the Fourier transform of each side:

$$\mathcal{F}\left\{\frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y\right\} = \mathcal{F}\left\{b_m \frac{d^m u}{dt^m} + \cdots + b_1 \frac{du}{dt} + b_0 u\right\}$$

$$\mathcal{F}\left\{\frac{d^n y}{dt^n}\right\} + \cdots + a_1 \mathcal{F}\left\{\frac{dy}{dt}\right\} + a_0 \mathcal{F}\{y\} = b_m \mathcal{F}\left\{\frac{d^m u}{dt^m}\right\} + \cdots + b_1 \mathcal{F}\left\{\frac{du}{dt}\right\} + b_0 \mathcal{F}\{u\}$$

$$(j\omega)^n Y(j\omega) + \cdots + a_1 (j\omega) Y(j\omega) + a_0 Y(j\omega) = b_m (j\omega)^m U(j\omega) + \cdots + b_1 (j\omega) U(j\omega) + b_0 U(j\omega)$$

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{U(j\omega)} = \frac{b_m (j\omega)^m + \cdots + j\omega b_1 + b_0}{(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \cdots + j\omega a_1 + a_0}}. \quad (*)$$

We call $H(j\omega)$ the frequency response function. It is a complex-valued function $H: \mathbb{R} \to \mathbb{C}$.

The following block diagrams are useful.

Frequency response function (Fourier): $\xrightarrow{U(j\omega)} \boxed{H(j\omega)} \xrightarrow{Y(j\omega)}$

Transfer function (Laplace): $\xrightarrow{U(s)} \boxed{H(s)} \xrightarrow{Y(s)}$

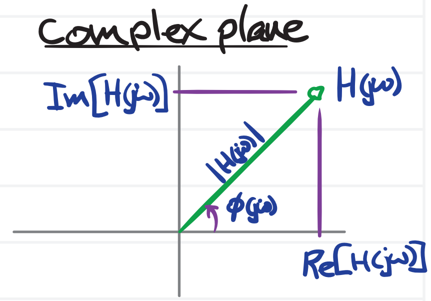
System transfer operator (time-domain): $\xrightarrow{u(t)} \boxed{H} \xrightarrow{y(t)}$

We can solve the SISO ODE by solving $(*)$ for $Y(j\omega)$ and taking the inverse F.T.:

$$\boxed{y(t) = \mathcal{F}^{-1}\{Y(j\omega)\} = \mathcal{F}^{-1}\{H(j\omega)\, U(j\omega)\}}.$$

We usually consider the magnitude $|H(\omega)|$ and phase $\phi(\omega)$, which are defined as

$$\boxed{\begin{aligned} |H(j\omega)| &\equiv \left(\mathrm{Re}[H(j\omega)]^2 + \mathrm{Im}[H(j\omega)]^2\right)^{1/2} \quad \& \\ \phi(j\omega) &\equiv \arctan \frac{\mathrm{Im}[H(j\omega)]}{\mathrm{Re}[H(j\omega)]} \end{aligned}}$$



Recall that we are only considering sinusoidal inputs. Assume

$$u(t) = A \sin(\omega t + \psi).$$

We have seen that the steady-state response is

$$\boxed{y(t) = \underbrace{A|H(\omega)|}_{\text{new amplitude}} \sin(\omega t + \underbrace{\psi + \phi(\omega)}_{\text{new phase}})}.$$

This was a very important result. It means that the steady-state response of a function to a sinusoidal input is sinusoidal and has a new amplitude and phase that depend on the frequency response function.

For this reason, the frequency response function's magnitude and phase give us significant insight into a system's s.s. response, and we like to plot them.

Bode plots: a review

Bode plots are how we typically plot $|H(j\omega)|$ and $\phi(j\omega)$.

First, recall that $X_{dB} \equiv 20 \log_{10} \frac{X_2}{X_1}$, where X_{dB} is a decibel representation of X_2, X_2 is an amplitude, and X_1 is a reference amplitude.

We will be plotting $|H(j\omega)|$ on a log-dB plot. Since $|H(j\omega)|$ is already a ratio (of output to input), we can simply write

$$|H(j\omega)|_{dB} = 20 \log_{10} |H(j\omega)| \quad .$$

Example Draw a bode plot for the system described by the SISO ODE

$$\tau \dot{y} + y = K u \quad .$$

Solution

Fourier transform: $\tau(j\omega)\,\mathcal{F}\{y\} + \mathcal{F}\{y\} = K\,\mathcal{F}\{u\}$

$$\Longrightarrow (j\omega\tau + 1)\,Y(j\omega) = K\,U(j\omega) \quad \Longrightarrow H(j\omega) \equiv \frac{Y(j\omega)}{U(j\omega)} = \frac{K}{1+j\omega\tau}$$

Now we have the frequency response function. We can find its magnitude and phase after we find its real & imaginary components.

$$H(j\omega) = \frac{K}{1+j\omega\tau} = \frac{K(1-j\omega\tau)}{(1-j\omega\tau)(1+j\omega\tau)} = \frac{K(1-j\omega\tau)}{1+(\omega\tau)^2}$$

so the components are $Re[H(j\omega)] = \dfrac{K}{1+(\omega\tau)^2}$ and $Im[H(j\omega)] = \dfrac{-\omega\tau K}{1+(\omega\tau)^2}$.

The magnitude and phase, then, are

$$|H(j\omega)| = \left(\left(\frac{K}{1+(\omega\tau)^2}\right)^2 + \left(\frac{-\omega\tau K}{1+(\omega\tau)^2}\right)^2\right)^{1/2}$$

$$= \left(\frac{K^2 + \omega^2\tau^2K^2}{(1+\omega^2\tau^2)^2}\right)^{1/2}$$

$$= \frac{K(1+\omega^2\tau^2)^{1/2}}{1+\omega^2\tau^2}$$

$$= \frac{K}{\sqrt{1+\omega^2\tau^2}}$$

$$\phi(j\omega) = \operatorname{atan} \frac{\dfrac{-\omega\tau K}{1+(\omega\tau)^2}}{\dfrac{K}{1+(\omega\tau)^2}}$$

$$= \operatorname{atan}(-\omega\tau)$$

$$= -\operatorname{atan}(\omega\tau)$$

The plots are shown below.