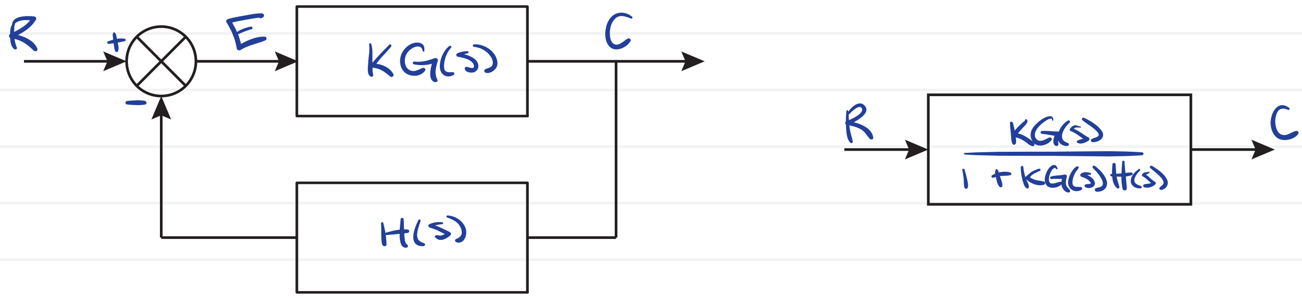
The Root Locus

Recall the control system problem of determining the closed-loop poles of a control system like the one shown.



The root locus is the "representation of the paths of the closed-loop poles as the gain [K] is varied" (Nise). It's actually fairly simple to solve for the roots of a polynomial numerically for a given value of K. The root locus is just the closed-loop pole plot for all values of K.

The closed-loop poles can be found by setting the denominator of the closed-loop transfer function to zero and solving for the values of s that satisfy this condition. Examining the closed-loop transfer function, we see that this is equivalent to

$$1 + KG(s)H(s) = 0 \implies KG(s)H(s) = -1 = \overbrace{1\angle(1+2k)\pi}^{\text{polar form}} \quad (k \in \mathbb{Z}).$$

$$\implies \boxed{\begin{array}{ll} \text{Magnitude criterion:} & |KG(s)H(s)| = 1 \quad \text{and} \\ \text{Angle criterion:} & \angle KG(s)H(s) = (1+2k)\pi. \end{array}}$$

The magnitude criterion implies that, for a point on the locus, the gain is

$$\boxed{K = \frac{1}{|G(s)|\,|H(s)|}} \;. \qquad (K \geq 0 \text{ in most cases})$$

This is the magnitude of a complex function: the reciprocal of G(s)H(s) to be precise. We can evaluate that function in the manner we learned previously, then:

$$\boxed{K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{open-loop pole magnitudes}}{\prod \text{open-loop zero magnitudes}}}.$$

This is an interesting result. In a moment we will be learning to sketch the root locus. This equation tells us that, given a desired closed-loop pole location $s = s_0$ on the root locus, we can compute the gain K required -- from evaluating the open-loop transfer function at s_0! This is a nice design tool.

Also note that K can be computed graphically because the magnitudes of the open-loop poles and zeros at some $s = s_0$ on the root locus are simply the lengths of the vectors from them to s_0.

Example

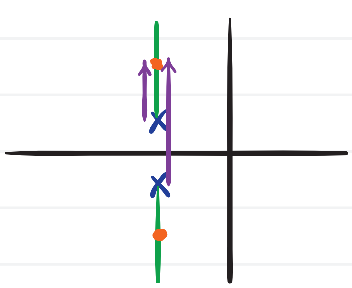
Given the following open-loop transfer function, plot the root locus in Matlab using the rlocus command (we will learn how to sketch it by hand in the next section, but for now we can just use Matlab).

$$KG(s)H(s) = K\frac{1}{s^2+2s+2}$$

If you were to place the poles $-1 \pm j\,3$, what gain K would you need?

Solution The open-loop poles are $p_{1,2} = -1 \pm \frac{1}{2}\sqrt{4-4\cdot 2}$
$= -1 \pm j1$.

There are no open-loop zeros. Let's plot them.



According to the Matlab, the root locus leaves the open-loop poles and goes straight up and down.

Drawing vectors from $p_{1,2} = -1 \pm j1$ to the desired closed-loop locations $-1 \pm j3$ gives $K = |(3-1)(3-(-1))| = 8$. //