02

A unique solution exists

We're not yet sure if a solution even *exists* for Equation 01.1, and if it does, if it is *unique*—meaning it's the only solution.

existence
uniqueness

02.01 Existence and uniqueness

Rather than proving the existence and uniqueness of a solution, we will simply consider a theorem that states conditions under which existence and uniqueness do hold. In other words: *a unique solution exists*, and we'll explore the conditions for which this is true.

Let the *forcing function* f be the "right-hand side" of Equation 01.1:

forcing function

$$f(t) \equiv b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \tag{02.1}$$

Theorem 02.1 (existence and uniqueness)**.** A solution $y(t)$ of Equation 01.1 *exists* and is *unique* for $t \geqslant t_0$ if and only if both the following are specified:

1. n initial conditions

$$y(t_0),\ \left.\frac{dy}{dt}\right|_{t=t_0}, \cdots, \left.\frac{d^{n-1}y}{dt^{n-1}}\right|_{t=t_0} \quad \text{and}$$

2. a continuous forcing function $f(t)$ for $t \geqslant t_0$.

Assuming this theorem can be proved, and it can (Finan, 2018), we need only the initial conditions and the forcing function to guarantee ourselves there is a unique solution. Let's think about what this means in terms of the dynamics of a system. In a sense, if we know its initial state and the input or forcing[1] how it will behave for the rest of time is determined. When I say "in a sense," I mean that insofar as the system is well-described by Equation 01.1. The determinist implications of this must be understood to be approximate and limited in scope. I don't want to be responsible for creating a bunch of determinists (Hoefer, 2016).

Note however, that, given initial conditions and forcing, we only know *that* a unique solution exists, not *what* that solution is or *how* to find it.

02.02 Outlining a solution technique

It turns out that, given a forcing function and no initial conditions, several potential solutions can satisfy the ODE Equation 01.1; conversely, given certain initial conditions and no forcing function, several potential solutions satisfy the ODE. It is only when both initial conditions and a forcing function are given that a unique solution exists. It can be shown (Kreyszig, 2010) that the *general solution* y_g (also called the *total solution*)—actually a "family" of solutions with unknown constants—to Equation 01.1 is equal to the sum of two solutions that are often relatively easy to obtain:

general solution y_g

1. the *homogeneous solution* y_h, another family of solutions, this time to Equation 01.1 with $f(t) = 0$, and
2. the *particular solution* y_p, which satisfies Equation 01.1 sans initial conditions.

homogeneous solution y_h

particular solution y_p

That is,

$$y_g(t) = y_h(t) + y_p(t). \tag{02.2}$$

Methods for deriving homogeneous and particular solutions are the topics of Lecture 03 and Lecture 04.

The general solution y_g is still a *family* of solutions that all satisfy Equation 01.1 for a given forcing function f. It only becomes the *unique* solution, which we call the *specific solution* and typically denote simply y or (occasionally) y_s, once the initial conditions are applied to y_g.

specific solution

The diagram of Figure 02.1 illustrates this solution technique, with each arrow signifying that the block at its tail is supplied to and precedes the block at its head. Lectures proceed with the diagram:

[1] Usually, if we know the input u, it is trivial to apply Equation 02.1 to find the forcing function f. However, note that u must be differentiable m times.

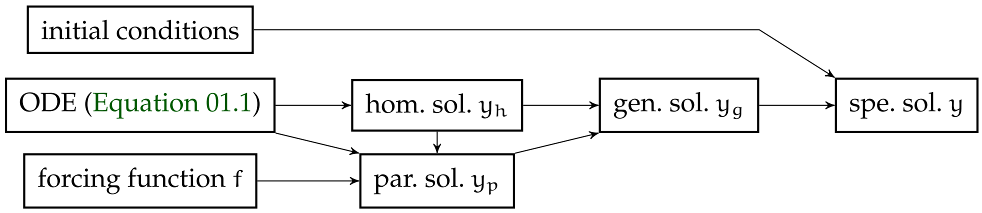

Figure 02.1: a diagram of the solution technique. Each arrow signifies that the block at its tail is supplied to and precedes the block at its head. The first column includes everything required to obtain a unique solution. The homogeneous y_h and particular y_p solutions of the second column sum to the general solution y_g. Applying the initial conditions to this yields the specific solution y.

- Lecture 03 describes how to obtain the homogeneous solution y_h from Equation 01.1 with the forcing function $f(t) = 0$;
- Lecture 04 describes how to derive the particular solution y_p from Equation 01.1 without the initial conditions for common forcing functions by a method called *undetermined coefficiencts*;
- Lecture 05 blows your mind by summing the homogeneous and particular solutions to obtain the general solution y_g; lest we be accused of dereliction of our duty to appear smarter than Business majors, this lecture also applies the initial conditions to the general solution to find constants introduced in the homogeneous solution to fully solve the differential equation—i.e., to obtain the specific solution y.

Box 02.1 Course connection: Differential Equations

The technique outlined here is probably quite similar to one described in your Differential Equations course. Terminology and notation may be different, so it may be worth correlating this primer with your previous coursework and text.