Lecture 01.01 Mathematical measurement theory

Mathematical measurement theory (sometimes just "measurement theory") concerns itself with how relations among mathematical objects (e.g. numbers) can represent relations among qualities of empirical objects (e.g. length). For instance, summing numbers is analogous to how the quality of length of a composite object is related to the quality of length of each of the object's constituents, as is illustrated in Figure 01.1.

But there's subtlety here that's best considered with precise language. Let's consider some fundamental definitions of measurement theory.

01.01.1 Quality, quantity, magnitude, and scale

When trying to describe what we are doing when we measure, there are certain terms are seemingly unavoidable. Therefore, it's worth considering some precise definitions of them. In the following, let's take the term *object* to mean the object of measurement.

object

Definition 01.01.1: quality

The *quality* of an object is the manner in which it interacts. It is the totality of its properties, which are aspects of the way the object interacts. (Spirkin, 1983)

We can think of the properties that constitute the quality of an object as sets to which a given object belongs or not, like "heavy" or "round." Immediately, however, we become suspicious that a real object can belong to such a set so completely or not. *Fuzzy set theory* allows members of a set to belong *to a certain degree* (Ross, 2010). Applying fuzzy set theory to measurement theory is beyond the scope of this text, but suffice it to say that the fuzziness of membership suggests a blurring of the boundary between quality and our next definition, quantity.

fuzzy set theory

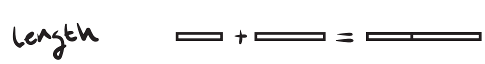

Figure 01.1: mathematical measurement theory explores the correspondence between mathematical objects like numbers and empirical qualities like lengths.

Definition 01.01.2: quantity

The *quantity* of an object is the amount of that which comprises it. The quantity of a finite collection of objects is the number of them. (Spirkin, 1983)

Note that we are already beginning to use mathematical analogy in our definition of quantity when we use a number to represent quantity. Quantity can be continuous or discrete. In the former case, it is often represented by a *real number*; in the latter, it is often represented by an *integer*.

real number

integer

We have used the term "amount" in the definition of the quantity of an object. This is a bit of a swindle, considering we have not yet defined this term. In fact, "amount" is not the usual scientific term. Rather, the term *physical magnitude* has emerged from physics (Hall, 2016). It is here that we bump against the limitation of language to define fundamental phenomena: physical magnitude is typically defined as a that which can be represented by a number about an object. Thus, we have interdependent definitions of quantity and physical magnitude when describing an object. We use both terms interchangeably.

physical magnitude

"I got a 20 on the exam." Without a measurement *scale*, there is no measurement.

Definition 01.01.3: measurement scale

A measurement *scale* is a mapping of quantities and qualities of an object to mathematical objects for representation. (Tal, 2017)

Measurement theory is hardly homogeneous, but we can think of it as being primarily comprised of considerations of (1) the nature of the objects of measurement and (2) the ways in which the correspondance between an object and its measure can be established. These are considered in the following sections (01.01.2, 01.01.3, and 01.01.4).

01.01.2 The nature of measurement objects

Every theory assumes an *ontology* (in the sense of metaphysics): a theoretical understanding of the nature of being. Unfortunately, we rarely consider ontology and instead thrash about with some assumed ontology—for there is no theory that does not have at least an implicit ontology. We will pause at ontology for just a moment before thrashing on.

ontology

There are several competing understandings of the ontological status of the objects of measurement. Tal (2017) describes them as

- concrete individual objects,
- qualitative observations of concrete individual objects,
- abstract representations of individual objets, and
- universal properties of objects.

This is especially important to realist theories of measurement, but is important to consider in all measurement theories.

01.01.3 Establishing scales

So, under mathematical measurement theory, quantities and qualities of an object are said to correspond in some way to mathematical objects. But how do we decide on the mathematical objects (scale)? What criteria are there for determining the efficacy of the mathematical objects?

This is another universal aspect of measurement theories: establishing the way in which scales can be properly established. It is a central consideration of most measurement theories.

01.01.4 Intrinsic and extrinsic quantities

intensive quantity

extensive quantity

Intensive quantities are those that represent properties of the constitutive substance of an object. Conversely, *extensive quantities* are those that are unique to each object.[2]

The quality of quantities (lol) to admit representation by a number leads to a simple manner in which to define the difference between an intrinsic and an extrinsic quantity: if an attribute of an object can be represented by the addition of numbers, it is an extrinsic quantity; otherwise, it is an intrinsic quality.[3] For instance, weight is best represented by a quantity because combining two objects with weights represented by w_1 and w_2 gives a composite object with weight $w_1 + w_2$. Similarly, the densities of two objects ρ_1 and ρ_2, if the objects are combined, are not.

[2] There's some ambiguity here, but this is approximately how the terms are used in the study of thermodynamics.

[3] The idea for this comes from (Campbell, 1920, p. 267), but his concepts of quantity and quality seem conflated with what we have called intrinsic and extrinsic quantities.

01.01.5 Fundamental and derived magnitudes

Early theories (Campbell, 1920) made the additivity and lack thereof of magnitudes the crucial aspect of a distinction between two types of physical magnitude: *fundamental* and *derived*.

fundamental magnitude
derived magnitude

Later theories distinguished between these two types of magnitude in the following way:

- *fundamental magnitudes* are those that can be measured directly and
- *derived magnitudes* are those that must be computed from a definition that depends on fundamental magnitudes.

For instance, density is measured by measuring both mass and volume of an object, then dividing them—making density a derived magnitude.

This means there is nothing intrinsic about the difference between a fundamental and a derived quantity; rather, a magnitude that is derived now may become fundamental if a method for measuring it directly is developed.

Example 01.01-1 fundamental versus derived

Of the following magnitudes, which is fundamental and which is derived?

1. resistance
2. length
3. mass
4. weight

There is an ambiguity here that I want to merely suggest and leave open. Let us take mass, for instance. We say it can be compared directly to another mass via a balance and therefore it is fundamental. However, how do we determine when a balance is balanced? By measuring, for instance, the angle of the balancing arm, which is surely never zero. It can only be "small enough." This hints at an issue with our conception of fundamental and derived quantities.

01.01.6 Classification of scales

Measurement scales have been classified by the types of transformation to which they are invariant without loss of empirical information. We will consider the following scales originated by Stevens (Tal, 2017; Robert, 1985).

nominal scales Nominal scales are those that are invariant to one-to-one substitution. Those that have no order are quintessential. For instance, gender or concave/convex (innie/outie) navels are nominal scales.

ordinal scales Ordinal scales are those that are invariant to monotonic, increasing transformations. Those that have a specific order are quintessential. For instance, one could feel *terribly*, *poorly*, or *meh*. Another example is physical hardness.

interval scales Interval scales are those that are invariant to positive linear transformation. Celcius and Farenheit scales for temperatures are related by just such a transformation

$$T_F = \frac{9}{5} \cdot T_C + 32 \tag{01.1}$$

without a loss in emperical information.

ratio scales Ratio scales are those that are invariant to multiplication by positive numbers. For instance, length can be represented in meters or kilometers via multiplication by a constant. Kelvin, unlike Celcius and Farenheit, is a ratio scale if negative Kelvin temperatures are excluded from consideration. Whenever a scale admits positive multiplication and excludes negative values (i.e. has an "absolute zero"), it is considered to be a ratio scale.

Example 01.01-2 scale classification

Classify the following measurement scales.

1. mass in kg
2. air quality index
3. numbered uniforms 1-99
4. time interval in sec
5. calendar time (e.g. 2017)

01.01.7 Representational theory

Representational Theory of Measurement

The *Representational Theory of Measurement* (RTM) is the most generally accepted mathematical measurement theory. It combines the considerations above—the nature of measurement objects and the classification of scales—to define measurement as "the construction of mappings from empirical relational structures into numerical relational structures" Tal (2017).

homomorphism

In this theory, measurement scales are *homomorphisms* (many-to-one mappings) from empirical relational structures to numerical relational structures.