Lecture 02.02 Fourier series

Fourier series are mathematical series that can represent a periodic signal as a sum of sinusoids at different amplitudes and frequencies. They are useful for solving for the response of a system to periodic inputs. However, they are probably most important *conceptually*: they are our gateway to thinking of signals in the *frequency domain*—that is, as functions of *frequency* (not time). To represent a function as a Fourier series is to *analyze* it as a sum of sinusoids at different frequencies ω_n and amplitudes and a_n. It's *frequency spectrum* is the functional representation of amplitudes a_n versus frequency ω_n.

frequency domain

Fourier analysis

frequency spectrum

Let's begin with the definition.

Definition 02.02.1: Fourier series: trigonometric form

The *Fourier analysis* of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and period T,

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(2\pi n t/T)\,dt \tag{02.9}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(2\pi n t/T)\,dt. \tag{02.10}$$

The *Fourier synthesis* of a periodic function $y(t)$ with analysis components a_j and b_j corresponding to ω_j is

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(2\pi n t/T) + b_n\sin(2\pi n t/T). \tag{02.11}$$

Let's consider the complex form of the Fourier series, which is analogous to Definition 02.02.

Definition 02.02.2: Fourier series: complex form

The *Fourier analysis* of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and period T,

$$c_{\pm n} = \frac{1}{T} \int_{-T/2}^{T/2} y(t) e^{-j2\pi n t/T} dt. \tag{02.12}$$

The *Fourier synthesis* of a periodic function $y(t)$ with analysis components c_n corresponding to ω_n is

$$y(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n t/T}. \tag{02.13}$$

We call the integer n a *harmonic* and the frequency associated with it

harmonic

$$\omega_n = 2\pi n/T \tag{02.14}$$

the *harmonic frequency*. There is a special name for the first harmonic ($n = 1$): the *fundamental frequency*. It is called this because all other frequency components are integer multiples of it.

harmonic frequency
fundamental frequency

It is also possible to convert between the two representations above.

Definition 02.02.3: Fourier series: converting between forms

The complex Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and a_n and b_n as defined above,

$$c_{\pm n} = \frac{1}{2}\left(a_{|n|} \mp j b_{|n|}\right) \tag{02.15}$$

The sinusoidal Fourier analysis of a periodic function $y(t)$ is, for $n \in \mathbb{N}_0$ and c_n as defined above,

$$a_n = c_n + c_{-n} \text{ and} \tag{02.16}$$

$$b_n = j\left(c_n - c_{-n}\right). \tag{02.17}$$

The *harmonic amplitude* is

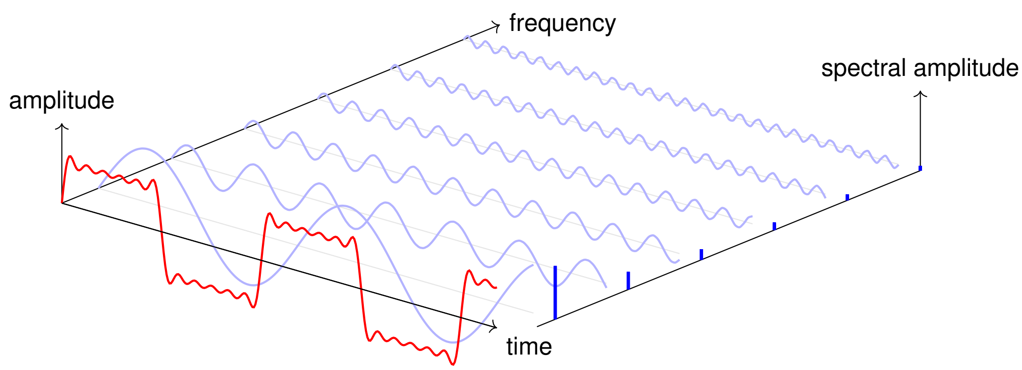
harmonic amplitude

$$C_n = \sqrt{a_n^2 + b_n^2} \tag{02.18}$$

$$= 2\sqrt{c_n c_{-n}}. \tag{02.19}$$

line spectrum A *line spectrum* is a graph of the harmonic amplitudes as a function of the harmonic frequencies.

The following illustration demonstrates how sinusoidal components sum to represent a square wave. A line spectrum is also shown.



Let us compute the associated spectral components in the following example.

Example 02.02-1 Fourier series analysis: line spectrum

Compute the first five harmonic amplitudes that represent the line spectrum for a square wave in the figure above.