Lecture 02.04 Sampling

While most quantities are continuous, making them naturally represented by continuous functions, in order to represent a signal in a computer, it must be given a *discrete* representation.[1] Constructing a discrete representation of a signal is called *sampling* it: a numerical value is assigned at discrete values of its domain. Since the quintessential signal has time as its domain, we will henceforth speak as if it is the only type.

discrete
sampling

Consider a function $f : \mathbb{R} \to \mathbb{R}$ of time t over the interval $[t_1, t_2]$ sampled at a constant interval T to form a sequence of reals $(f_n)_{n \in \mathbb{N}_0}$ of length $N = 1 + (t_2 - t_1)/T$ called the *sampled sequence*, defined as follows.

sampled sequence

Equation 02.27 sampled sequence values

We represent the sampling process as the multiplication of the continuous function f by the *Dirac comb function* $s : \mathbb{R} \to \mathbb{R}$ defined as

Dirac comb function

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \tag{02.28}$$

where δ is the Dirac delta function, and illustrated in Figure 02.3. This sampling representation yields the *sampled function* $f^\star : \mathbb{R} \to \mathbb{R}$ defined as follows.

sampled function

Equation 02.29 sampled function

This representation requires some interptretation. Recall that δ is zero everywhere in its domain except at $t = 0$, when it is *undefined*, but has an integral over the "pulse" of unity. Further recall that *scaling* δ by some factor λ yields the same functional value, but it is understood in the *distribution* sense to be multiplying the integral over the pulse by λ. We call this the pulse *strength*. So the sample sequence values f_n are computed from $f^\star$ by

δ strength

[1]Good references for the sampling and related topics are Rowell (2008) and Gene F. Franklin (1998).

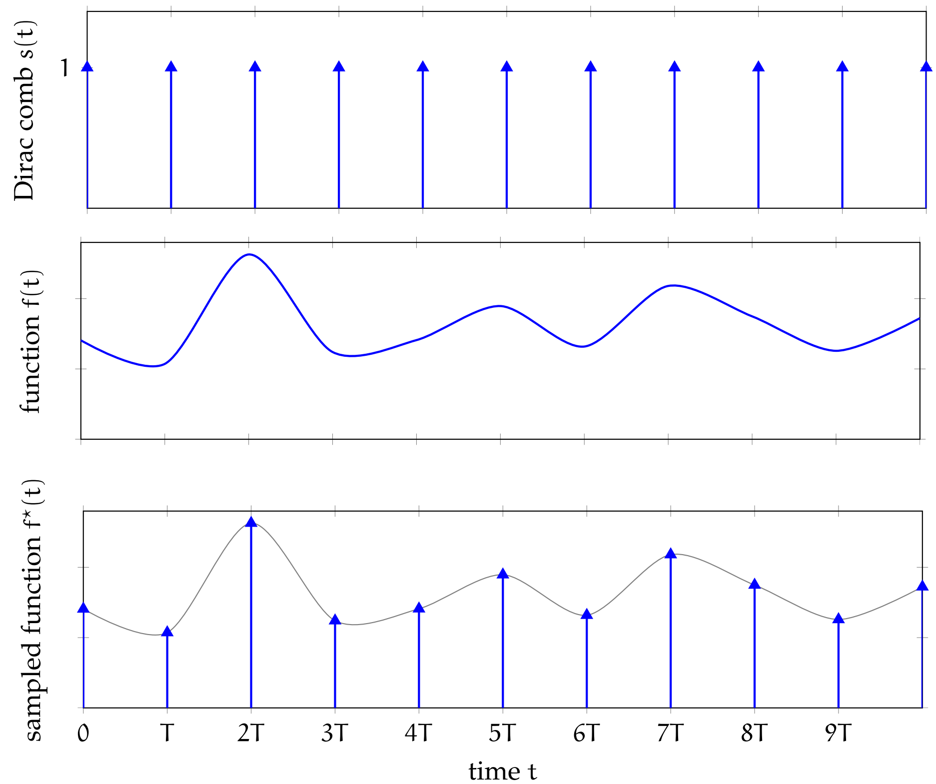

Figure 02.3: (top) the Dirac comb function s of Equation 02.28, (middle) a function f to be sampled, and (bottom) a sampled function f^*. The height of each pulse is f_n and represents the strength of the pulse.

the following equation.

Equation 02.30 sampled sequence values from f^*

In other words, we integrate over the n^{th} pulse to get the sample f_n.

02.04.1 Spectrum of a sampled function

The Dirac comb function s is periodic and therefore has a Fourier series representation. From Definition 02.02, we can compute the components sifting property

The Dirac delta function has a nice property called the *sifting property* that states that for some function g, the integral $\int_I \delta(t-\tau)g(t)dt = g(\tau)$ over the interval I and 0 otherwise. This yields

We happend to choose the easiest interval over which to integrate, but the same holds for any other period, which means $c_{\pm n} = 1/T$ for all time. The Fourier series synthesis, then, is

The spectrum $F^*(\omega)$ of a sampled function f^* can be found directly from the definition of the Fourier transform (**??**)

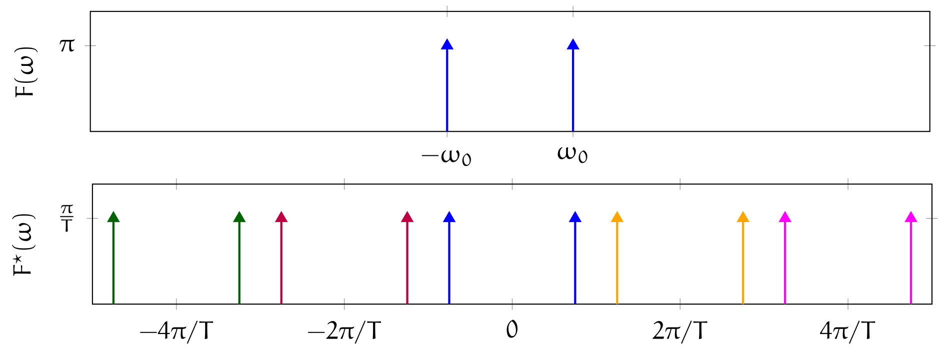

Figure 02.4: (top) the Fourier transform F of function $f(t) = \cos\omega_0 t$ and (bottom) the Fourier transform $F^\star$ of sampled function $f^\star$. T is the *sampling period*.

This result means the Fourier transform $F^\star$ of the sampled function $f^\star$ is a periodic repetition (with frequency-domain period $2\pi/T$) of the Fourier transform F of the continuous signal f, scaled by $1/T$.

For instance, although it is periodic and so has a trivial Fourier series, the cosine function $f(t) = \cos\omega_0 t$ also has Fourier transform

$$F(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0). \tag{02.31}$$

This is illustrated in the top spectrum of Figure 02.4. Our expression above for $F^\star$ in terms of F allow us to construct the spectrum for $F^\star$, shown in the bottom spectrum of Figure 02.4.