Lecture 02.05 Nyquist sampling theorem, aliasing, and reconstruction

When we sample a signal, we lose information. However, we can reconstruct an approximation of the original signal if we make some assumptions about it.

Theorem 02.05.1: Nyquist sampling theorem

If a signal has maximum frequency component f_{max}, if it is sampled at a frequency greater than

$$f_N = 2f_{max} \tag{02.32}$$

it can be *unambiguously reconstructed* and otherwise not. We call f_N the *Nyquist frequency* of a signal.

aliasing

The "not" case is characterized by *aliasing*: when a signal appears to have frequency components that are in fact higher than they appear—an artifact of sampling.

This is the first time in our journey that Fourier analysis and frequency domain concepts will light the path. There are several ways to analyze aliasing, but perhaps considering the spectrum of a sampled sinusoid is the most accessible.

Let us consider, as in Figure 02.4, the sampling of the function $y(t) = \cos\omega_0 t$, where $\omega_0 \in \mathbb{R}$. The signal has the Fourier transform

which is just a pulse with strength π at $\omega = -\omega_0$ and one at $\omega = +\omega_0$. The Fourier transform Y^* of the sampled signal y^* simply "copies" this pair of pulses to be mirrored about integer multiples of the sampling angular frequency ω_s.

According to **??**, $\omega_s > \omega_N = 2\pi f_N$ is required to avoid aliasing and allow us to uniquely reconstruct the original signal. Let's consider the situation in which the Nyquist sampling frequency requirements are met, as shown in the upper plot of Figure 02.5. Now, consider the case for which the Nyquist sampling requirements are not met and there is aliasing, as shown in the lower plot of Figure 02.5. These two spectra are

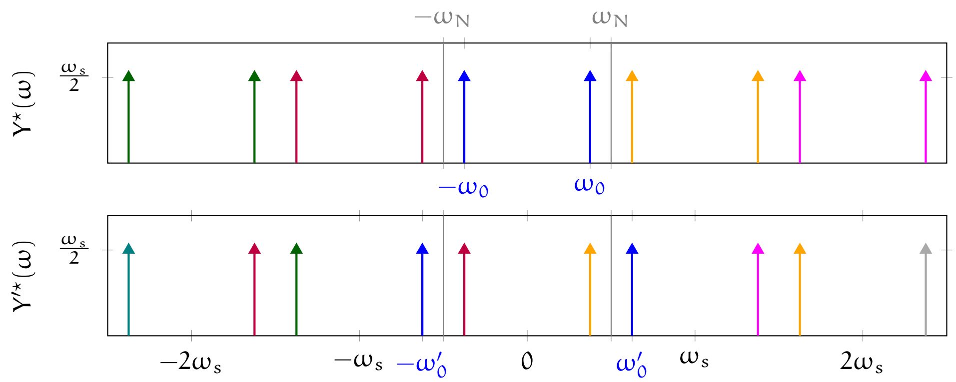

Figure 02.5: (top) the Fourier transform Y^* of sampled function $y^*(t) = \cos\omega_0 t$ and (bottom) the Fourier transform Y'^* of sampled function $y'^* = \cos\omega_0' t$, where the Nyquist sampling frequency ω_N is between them: $\omega_0 < \omega_N < \omega_0'$. That is, y^* is sufficiently sampled but y'^* is *not*, and is therefore aliased. The two spectra are *indistinguishable* and therefore one must be confident *a priori* that the sampling frequency is greater than ω_N.

indistinguishable and therefore if we simply assume that the component at ω_0 is "real" (i.e. not aliased), we might be mistaken.

It is important to note that *once the signal is sampled, it's too late to do anything about it*. There are two ways to mitigate aliasing:

1. sample at a high enough rate to capture all frequency components of the signal and
2. *low-pass* or *anti-aliasing filter* the signal *before* it is sampled.

anti-aliasing filter

The former always begs the question—it is never known if the sample rate is high enough. The latter is always advisable, although it only minimizes the effects of aliasing.

Assuming there is no aliasing (which is never more than approximately true), a continuous signal y can be fully reconstructed from its sample sequence (y_n) for N samples and sample period T by the *Whittaker–Shannon interpolation formula* Rowell (2008)

Whittaker–Shannon interpolation formula

$$y(t) = \sum_{n=0}^{N-1} y_n \frac{\sin(\pi(t - nT)/T)}{\pi(t - nT)/T}. \tag{02.33}$$