Lecture 03.01 Dynamic system representations

graphical representations
mathematical representations

Dynamic systems—measurement and otherwise—have several representations, mostly *graphical* or *mathematical*. For instance, schematics, linear graphs, and block diagrams are graphical representations. The foci of this lecture are three mathematical representations and a graphical representation:

1. the input-output, linear ordinary differential equation,
2. the transfer function,
3. the frequency response function, and
4. the block diagram.

Refer to Figure 03.1 for an illustration of the relations among system representations.

03.01.1 Input-output ordinary differential equations

Consider a dynamic system described by the *input-output differential equation*—with independent variable $y(t)$ (called the *output*), dependent variable time t, *input* $u(t)$, constant coefficients a_i, b_j, order n, and $m \leqslant n$ for $n \in \mathbb{N}_0$—as:

$$\begin{aligned}\frac{d^n y}{dt^n} + \quad a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = \\ b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u.\end{aligned} \tag{03.1}$$

single-input, single output (SISO

This might describe, for instance, an output voltage reading from an input force on a piezo-electric transducer. Mechanical, electronic, thermal, fluidic, and many other types of dynamic systems can be described with Equation 03.2. It is important to note that this describes the relationship between a *single-input* and a *single-output (SISO)*. However, a great many measurement systems behave approximately like linear SISO systems (at least in some operating regimes).

03.01.2 Transfer functions

Consider a dynamic system described by the *input-output differential equation*—with variable y representing the *output*, dependent variable time t, variable u representing the *input*, constant coefficients a_i, b_j, order n, and

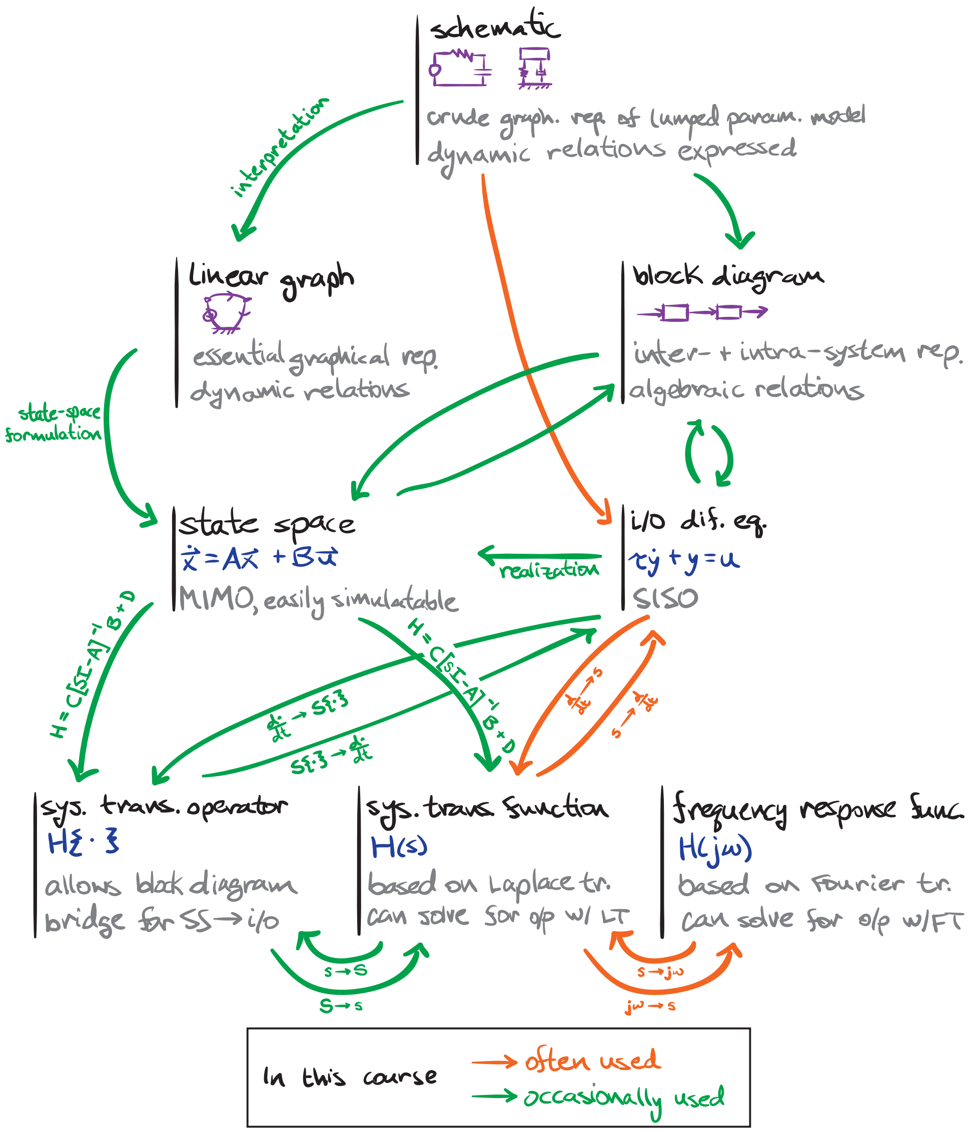

Figure 03.1: relations among system representations.

$m \leqslant n$ for $n \in \mathbb{N}_0$—as:

$$\begin{aligned} \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = \\ b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \end{aligned} \tag{03.2}$$

Laplace transform

The *Laplace transform* $\mathcal{L}$ of Equation 03.2 yields something interesting (assuming zero initial conditions):

$$\begin{aligned}
&\mathcal{L}\left(\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y\right) = \\
&\mathcal{L}\left(b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u\right) \quad \Rightarrow \\
&\mathcal{L}\left(\frac{d^n y}{dt^n}\right) + a_{n-1}\mathcal{L}\left(\frac{d^{n-1} y}{dt^{n-1}}\right) + \cdots + a_1 \mathcal{L}\left(\frac{dy}{dt}\right) + a_0 \mathcal{L}(y) = \\
&b_m \mathcal{L}\left(\frac{d^m u}{dt^m}\right) + b_{m-1}\mathcal{L}\left(\frac{d^{m-1} u}{dt^{m-1}}\right) + \cdots + b_1 \mathcal{L}\left(\frac{du}{dt}\right) + b_0 \mathcal{L}(u) \quad \Rightarrow \\
&s^n Y + a_{n-1} s^{n-1} Y + \cdots + a_1 s Y + a_0 Y = \\
&b_m s^m U + b_{m-1} s^{m-1} U + \cdots + b_1 s U + b_0 U.
\end{aligned}$$

Solving for Y,

forced response

The inverse Laplace transform $\mathcal{L}^{-1}$ of Y is the *forced response*. However, this is not our primary concern; rather, we are interested to solve for the transfer function H as the ratio of the output transform Y to the input transform U, i.e.

$$H(s) \equiv \frac{Y(s)}{U(s)} \tag{03.3}$$

$$= \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}. \tag{03.4}$$

block diagram

This is the foundation of another graphical technique called *block diagrams*.

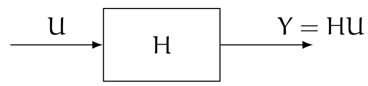

Figure 03.2: a block diagram for transfer function $H(s)$ from input $U(s)$ to output $Y(s)$.

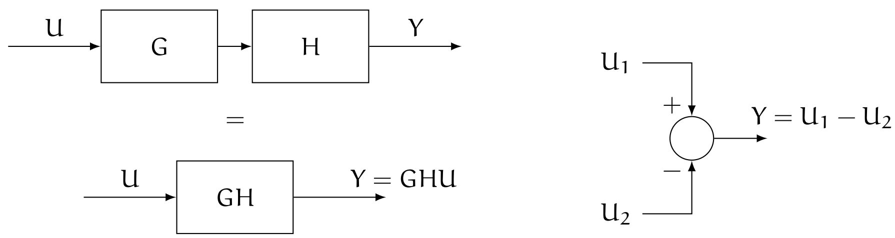

Figure 03.3: block diagrams showing (left) concatenation (i.e. multiplication) of G and H and (right) summation of inputs U_1 and U_2 (actually, subtraction).

03.01.3 Block diagrams

Block diagrams (also called *operational block diagrams*) represent inter- and intra-system relationships. A *block*, shown in Figure 03.2, represents a dynamic system with input $U(s)$ and output $Y(s)$ and (typically) transfer function $H(s)$.

block

Each block can be considered, in the Laplace domain, to *multiply* its input by the transfer function, as illustrated in Figure 03.3 (left). Also shown in Figure 03.3 (right) is the *summation block* or *junction* that adds (or subtracts) its inputs.

multiplication

summation block

Block diagrams differ fundamentally from circuit diagrams and linear graphs in that do not "load" each other. That is, concatenating two systems does not affect the operation of the first. However, since we can write an input-output differential equation for a circuit and we can write a transfer function for a differential equation—block diagrams can represent a circuit or many other dynamic systems. We must be careful to make sure that subsequent connections do not load the output of another block. That is, *blocks have infinite input impedance*.

03.01.4 Frequency response functions

Let a system have input u and output y. We define its *frequency response function* $H(j\omega)$ to be the ratio of the Fourier transform its output to the Fourier transform of its input:

frequency response function

Now, we have used a strange notation for the Fourier transform of a function, including the $j = \sqrt{-1}$ in the argument. In the context of frequency response, this is both conventional and gross. We will see why it's (barely) useful in a moment.

Typically, a system's dynamical model will first be developed as a differential equation—for instance, our input-output differential equation. We have already learned how to find a system's transfer function $H(s)$—which is the *Laplace* transform of the ratio of input to output. A shortcut to deriving the frequency response function $H(j\omega)$ from the transfer function $H(s)$ is to note that the Fourier transform of a function is equal to the Laplace transform of that function with the substitution $s \to j\omega$. For the frequency response function,

$$H(j\omega) = H(s)|_{s \to j\omega}. \tag{03.7}$$

gauche af

Note that we've used the same symbol H for both the Fourier and Laplace transforms, which is *gauche af*, but which is somewhat mitigated by our other violation of good taste: the argument s goes with the Laplace transform and the argument $j\omega$ goes with the Fourier transform.

meaning of $H(j\omega)$

So what does the frequency response function *mean*? It describes, in the frequency domain, how a system's input relates to its output. In Lecture 03.09, we will consider this interpretation in greater detail. Finally, note that the steady-state solution can be had by taking the inverse Fourier transform:

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\left(Y(j\omega)\right) \\ &= \mathcal{F}^{-1}\left(H(j\omega)U(j\omega)\right). \end{aligned} \tag{03.8}$$

Due to its greater generality (i.e. broader applicability), we typically prefer the inverse Laplace transform (and transfer function) for this task.

Example 03.01-1 converting representations

For the input-output differential equation

$$2\frac{d^2y}{dt} + 4\frac{dy}{dt} + 6y = 10\frac{du}{dt} + 2u.$$

find the transfer function $H(s)$ and the frequency response function $H(j\omega)$.