Lecture 03.03 First-order measurement systems

First order measurement systems have input-output differential equations of the form

$$\tau \frac{dy}{dt} + y = b_1 \frac{du}{dt} + b_0 u \tag{03.10}$$

with $\tau \in \mathbb{R}$ called the *time constant* of the system. Measurement systems with a single energy storage element—such as those with electrical or thermal capacitance—can be modeled with first-order systems.

time constant

03.03.1 Step response

Commonly, a scaling of the *unit step function* $u_s(t)$, which is 0 for $t < 0$ and 1 for $t \geqslant 0$, can be considered the input to this and other measurement systems (e.g. whenever the input is changed suddenly, u_s is a good approximation). If we consider the common situation that $b_1 = 0$ and $u(t) = K u_s(t)$ for some $K \in \mathbb{R}$, the solution to Equation 03.10 is

unit step function

If we assume the steady-state solution is the proper measurement value, the transient response is *error*. Considering it never reaches zero in finite

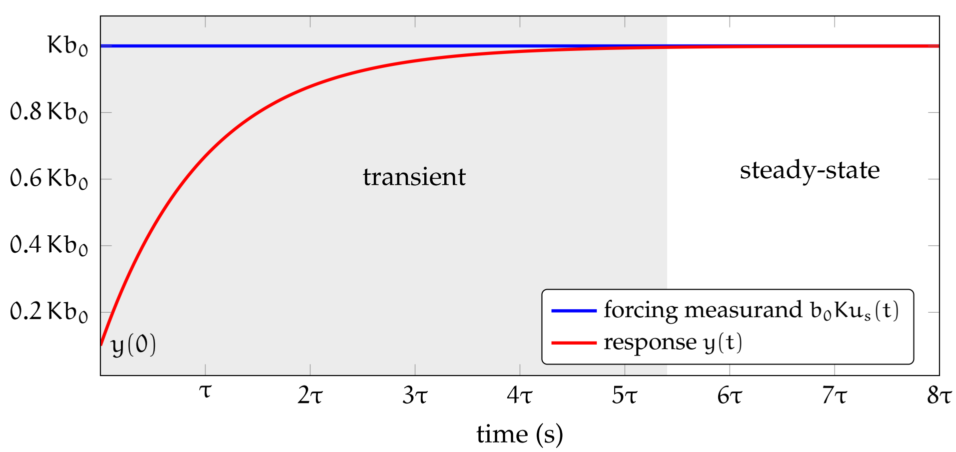

Figure 03.4: (step) response $y(t)$ of a first order system with input $u(t) = K u_s(t)$ and $b_1 = 0$.

time, this is a bummer ☹. However, it does decay exponentially, so in 5τ, the transient response is less than 1 % of difference between $y(0)$ and steady-state. A plot of the step response is shown in Figure 03.4.

03.03.2 Sinusoidal response

Another common input to first-order measurement systems modeled by Equation 03.10 is the sinusoid $u(t) = A \sin \omega t$. For $b_1 = 0$, the solution is

where κ can be found from the initial condition $y(0)$ to be

$$\kappa = y(0) + \frac{b_0 A}{\sqrt{1 + (\omega\tau)^2}} \sin(\arctan(\omega\tau)). \tag{03.11}$$

Figure 03.5 shows responses of a first-order measurement system to sinusoidal inputs (measurands) at different frequencies ω. Note the transient response decays with the same rate τ no matter the input frequency. However, there are two differences in the steady-state responses: the amplitude and phase. In fact, the steady-state amplitude and phase of an output compared to an input present a form of *error* in the measurement. Ideally, the ratio of the output and input is unity; however, for positive ω, this is never quite the case. We define this ratio, called the *magnitude ratio* $M(\omega)$, to be the steady-state output amplitude over the forcing amplitude. For first-order systems,

magnitude ratio $M(\omega)$

A metric for the nearness of $M(\omega)$ to unity is called the *dynamic error* $\delta(\omega)$, given by

dynamic error

Ideally, $\delta(\omega) = 0$, but as we can see from the expression for $M(\omega)$, this is only ever approximately true for nonzero τ and ω. However, it is small when the product $\omega\tau$ is small. So, in order to minimize dynamic

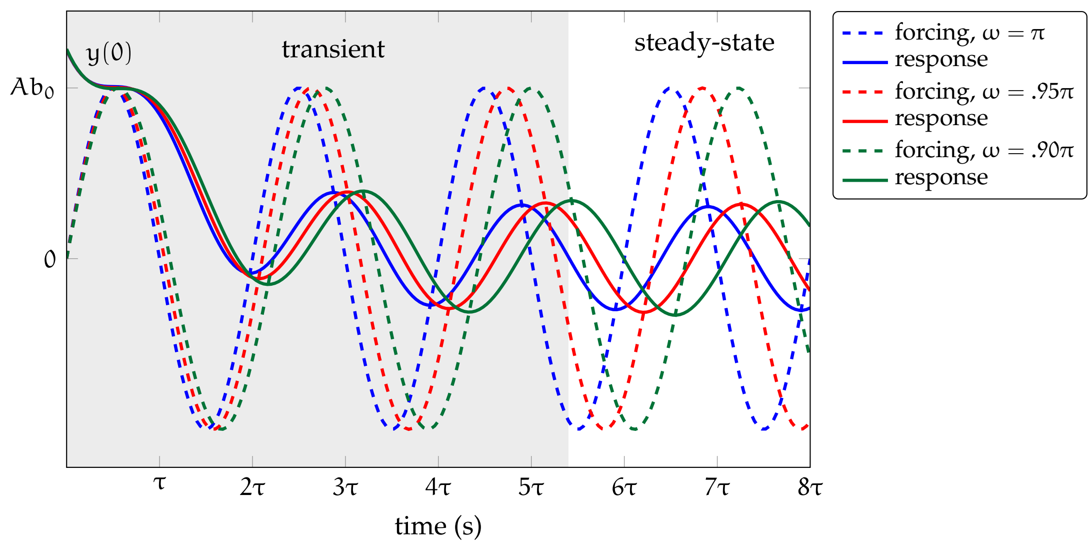

Figure 03.5: response $y(t)$ of a first order system with input $u(t) = A \sin \omega t$ and $b_1 = 0$ for three values of ω. The forcing function (measurand) is $b_0 A \sin \omega t$.

error for the measurement of a sinusoid at a given frequency, we must strive to minimize the time constant τ. It is common to call "good enough" $M(\omega) \geqslant .707$.

Similarly, the phase difference of the output relative to the input is ideally zero. Therefore, the phase shift $\phi(\omega)$ is another type of error and, for first-order systems, is given by

This corresponds to a *time-delay* $\beta_1(\omega)$ in the measurement:

time-delay $\beta_1(\omega)$

Clearly, we want to minimize $\phi(\omega)$ and $\beta_1(\omega)$. Typically, this is achieved by minimizing τ, which corresponds to the minimization of τ for the minimization of the dynamic error.

Note that the steady-state response of the measurement system to sinusoidal inputs is characterized by $M(\omega)$ and $\phi(\omega)$. In fact, a crucial identify will be observed here:

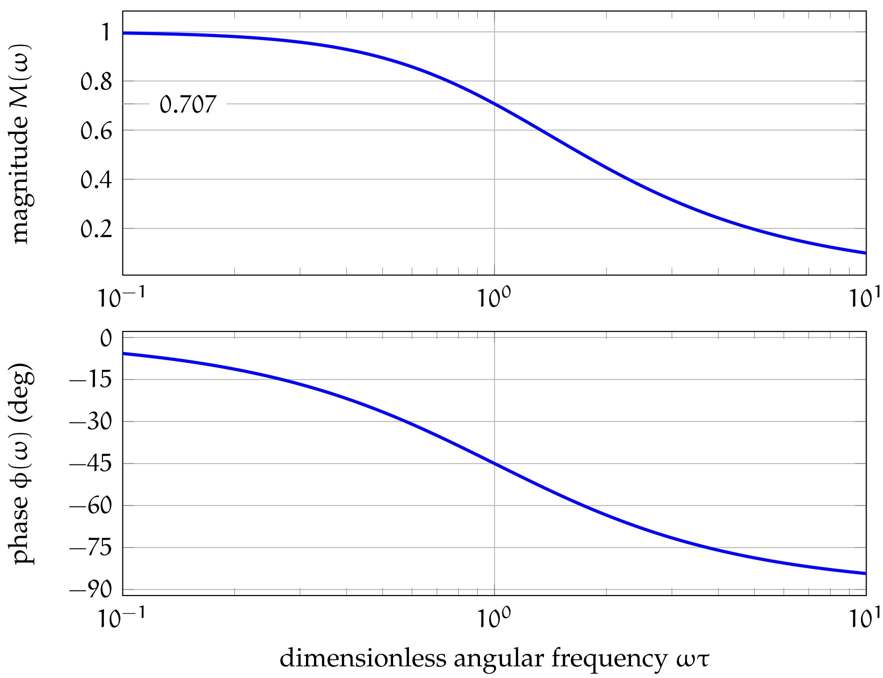

Figure 03.6: the magnitude ratio and phase.

> *the magnitude ratio* $M(\omega)$ *and phase* $\phi(\omega)$ *are equal to the magnitude* $|H(j\omega)|$ *and phase* $\angle H(j\omega)$ *of the frequency response function* $H(j\omega)$.

This is recognized as being the complex amplitude of the output over the input, which are plotted in Figure 03.6.