Lecture 03.04 Second-order measurement systems: free response

Second-order measurement systems have input-output differential equations of the form

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = f(t) \tag{03.12}$$

where ω_n is called the *natural frequency*, ζ is called the (dimensionless) *damping ratio*, and f is a forcing function that depends the input u as

natural frequency ω_n
damping ratio ζ

$$f(t) = b_2 \frac{d^2u}{dt^2} + b_1 \frac{du}{dt} + b_0 u. \tag{03.13}$$

Measurement systems with two energy storage elements—such as those with an inertial element and a spring-like element—can be modeled with second-order systems.

For distinct roots ($\lambda_1 \neq \lambda_2$), the homogeneous solution is, for some real constants κ_1 and κ_2,

$$y_h(t) = \kappa_1 e^{\lambda_1 t} + \kappa_2 e^{\lambda_2 t} \tag{03.14}$$

where

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}. \tag{03.15}$$

03.04.1 Free response

The *free response* y_{fr} of a system is its response to initial conditions and no forcing ($f(t) = 0$). This is useful for two reasons:

free response y_{fr}

1. perturbations of the measurement system from equilibrium result in free response, making it critical; and
2. the free response can be added to a forced response.

The free response is found by applying initial conditions to the homogeneous solution. With initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$, the free response is

$$y_{fr}(t) = y_0 \frac{1}{\lambda_2 - \lambda_1} \left(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right). \tag{03.16}$$

There are five possibilities for the location of the roots λ_1 and λ_2, all determined by the value of ζ.

$\zeta \in (-\infty, 0)$: **unstable** This case is very undesirable because it means our measurement system is unstable and, given any nonzero input or output, will *explode* ☹ to infinity. Not a good look.

boom ☹

$\zeta = 0$: **undamped** An undamped system will oscillate forever if perturbed from zero output. Once again, a bad look for a measurement device.

$\zeta \in (0, 1)$: **underdamped** Roughly speaking, underdamped systems oscillate, but not forever. Let's consider the form of the solution for initial conditions and no forcing. The roots of the characteristic equation are

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d \tag{03.17}$$

damped natural frequency ω_d

where the *damped natural frequency* ω_d is defined as

$$\omega_d \equiv \omega_n\sqrt{1-\zeta^2}. \tag{03.18}$$

From Equation (03.16) for the free response, using Euler's formulas to write in terms of trigonometric functions, and the initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$, we have

$$y_{fr}(t) = y_0\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \psi) \tag{03.19}$$

where the phase ψ is

$$\psi = \arctan\frac{\zeta}{\sqrt{1-\zeta^2}}. \tag{03.20}$$

This is an oscillation that decays to the value it oscillates about, $y(t)|_{t\to\infty} = 0$. So any perturbation of a critically damped measurement system will result in a decaying oscillation about equilibrium.

$\zeta = 1$: **critically damped** In this case, the roots of the characteristic equation are equal:

$$\lambda_1 = \lambda_2 = -\omega_n \tag{03.21}$$

So we must modify Equation 03.14 with a factor of t for the homogeneous solution. The free response for initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$, we have

$$y_{fr}(t) = y_0\,(1+\omega_n t)\,e^{-\omega_n t}. \tag{03.22}$$

This decays without oscillation, but just barely.

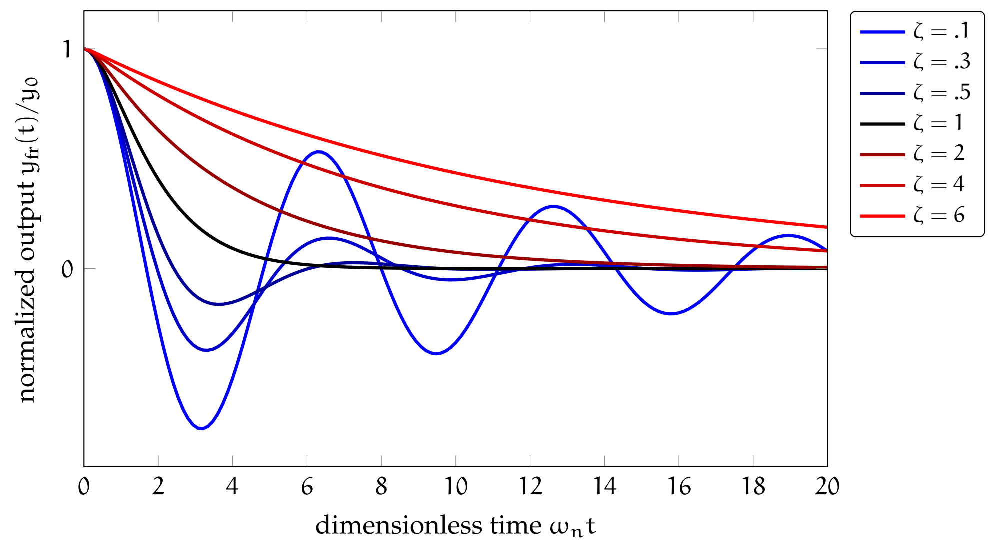

Figure 03.7: free response $y_{fr}(t)$ of a second-order system with initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$ for different values of ζ. Underdamped, critically damped, and overdamped cases are displayed.

$\zeta \in (1, \infty)$: **overdamped** Here the roots of the characteristic equation are distinct and real. From Equation (03.16) with free response to initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$, we have the sum of two decaying real exponentials. This response will neither overshoot nor oscillate—like the critically damped case—but with even lesser gusto.

Figure 03.7 displays the free response results. Note that a small damping ratio results in overshooting and oscillation about the equilibrium value. In contrast, large damping ratio results in neither overshoot nor oscillation. However, both small and large damping ratios yield responses that take longer durations to approach equilibrium than damping ratios near unity. For this reason, the damping ratio of a measurement system should be close to one. There are tradeoffs on either side of $\zeta = 1$. Slightly less than one yields faster responses that overshoot a small amount. Slightly greater than one yields slower responses less prone to oscillation.

Example 03.04-1 MRFM cantilever beam detector

In magnetic resonance force microscopy (MRFM), the primary detector is a small cantilever beam with a magnetic tip. Model the beam as a spring-mass-damper system with mass $m = 6$ pg,[a] spring constant $k = 15$ mN/m, and damping coefficient $B = 37.7 \cdot 10^{-15}$ N·s/m.

1. What is the natural frequency ω_n?
2. What is the damping ratio ζ?
3. In free response, how long before the amplitude must be less than 10% of its initial value? An upper bound is sufficient.

[a] One pg $= 10^{-12}$g $= 10^{-15}$kg.