Lecture 03.05 Second-order measurement systems: forced response

Second-order measurement systems are subjected to a variety of forcing functions f. In this lecture, we examine two common varieties: step forcing and sinusoidal forcing. In what follows, we develop *forced response* y_{fo} solutions, which are the *specific solution* responses of systems to given inputs and *zero initial conditions*: all initial conditions set to zero. In Lecture 03.08, a method is presented for combining free and forced response.

forced response y_{fo}

zero initial conditions

03.05.1 Step response

Step forcing of the form $f(t) = Ku_s(t)$, where $K \in \mathbb{R}$ and u_s is the unit step function, models abrupt changes to the input (measurand). The solution is found by applying zero initial conditions ($y(0) = 0$ and $\dot{y}(0) = 0$) to the general solution. If the roots of the characteristic equation λ_1 and λ_2 are distinct, the forced response is

$$y_{fo}(t) = \frac{K}{\omega_n^2}\left(1 - \frac{1}{\lambda_2 - \lambda_1}\left(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right)\right) \tag{03.23}$$

where

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{03.24}$$

Once again, there are five possibilities for the location of the roots of the characteristic equation λ_1 and λ_2, all determined by the value of ζ. However, there are three important cases for measurement systems: underdamped, critically damped, and overdamped.

$\zeta \in (0, 1)$ **underdamped** In this case, the roots are distinct and complex:

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm j\omega_d. \tag{03.25}$$

From Equation 03.23, the forced step response is

$$y_{fo}(t) = \frac{K}{\omega_n^2}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\cos(\omega_d t + \psi)\right) \tag{03.26}$$

where the phase ψ is

$$\psi = \arctan\frac{\zeta}{\sqrt{1 - \zeta^2}}. \tag{03.27}$$

This response overshoots, oscillates about, and decays to K/ω_n^2.

$\zeta = 1$ **critically damped** The roots are equal and real:

$$\lambda_1, \lambda_2 = -\omega_n \tag{03.28}$$

so the forced step of Equation 03.23 must be modified; it reduces to

$$y_{fo}(t) = \frac{K}{\omega_n^2}\left(1 - e^{-\omega_n t}(1 + \omega_n t)\right). \tag{03.29}$$

This response neither oscillates nor overshoots its steady-state of $\frac{K}{\omega_n^2}$, but just barely.

$\zeta \in (1, \infty)$ **overdamped** In this case, the roots are distinct and real, given by Equation 03.24. The forced step given by Equation 03.23 is the sum of two decaying real exponentials. These responses neither overshoot nor oscillate about their steady-state of K/ω_n^2. With increasing ζ, approach to steady-state slows.

Figure 03.8 displays the forced step response results. These responses are inverted versions of the free responses of 03.04.1. Note that a small damping ratio results in overshooting and oscillation about the steady-state value. In contrast, large damping ratio results in neither overshoot nor oscillation. However, both small and large damping ratios yield responses that take longer durations to approach equilibrium than damping ratios near unity. For this reason, the damping ratio of a measurement system should be close to $\zeta = 1$. There are tradeoffs on either side of one. Slightly less yields faster responses that overshoot a small amount. Slightly greater than one yields slower responses less prone to oscillation.

03.05.2 Sinusoidal response

Here we consider only steady-state sinusoidal response, allowing us to focus on frequency-domain considerations. The second-order system transfer function, found from the Laplace transform of Equation 03.12, from input u (generating forcing function $f(t) = Ku(t)$) to output y has the form

$$H(s) = \frac{K/\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{03.30}$$

The frequency response function $H(j\omega)$ is found via the substitution $s \to j\omega$, where ω is the input sinusoidal frequency:

$$H(j\omega) = \frac{K/\omega_n^2}{(1 - (\omega/\omega_n)^2) + j(2\zeta\omega/\omega_n)}. \tag{03.31}$$

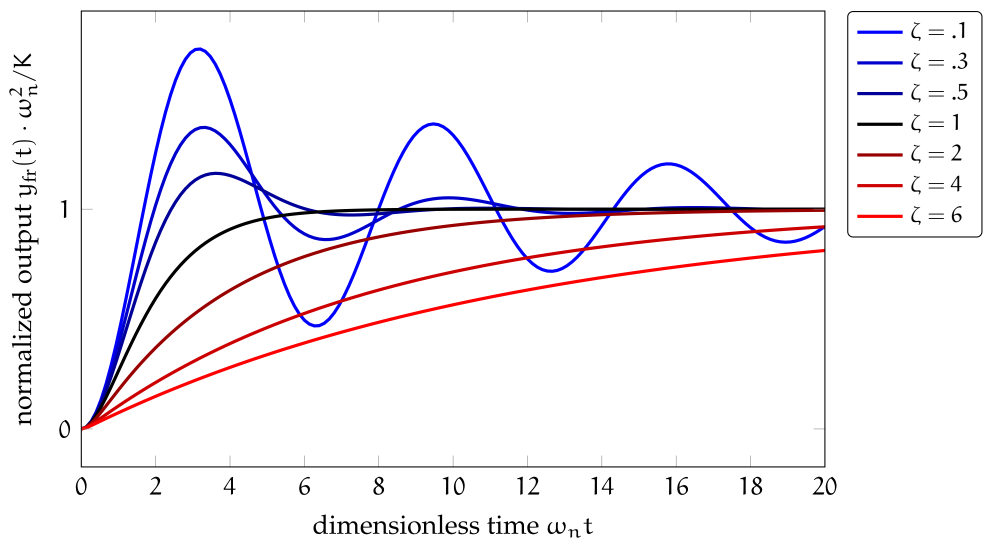

Figure 03.8: forced step response $y_{fo}(t)$ of a second-order system for different values of ζ. Underdamped, critically damped, and overdamped cases are displayed.

Writing this in terms of a magnitude and phase,

$$|H(j\omega)| = \frac{K/\omega_n^2}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2}} \quad \text{and} \tag{03.32a}$$

$$\angle H(j\omega) = \arctan \frac{-2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}. \tag{03.32b}$$

These functions are plotted in Figure 03.9 for a range of ζ. Note especially that the magnitude $|H(j\omega)|$ is near unity for low frequency, peaks (for underdamped systems) near ω_n, and tapers to zero high frequency. This corresponds to amplitude ratios between the input sinusoidal amplitude and output sinusoidal amplitude.

The phase $\angle H(j\omega)$ is near zero for low frequency is -90 deg at ω_n, and approaches -180 deg for high frequency. This corresponds to a phase lag between the input and output sinusoids.

For input $u(t) = A\sin(\omega t + \phi)$, the steady-state response y_{ss} can be found directly from the frequency response:

$$y_{ss}(t) = A|H(j\omega)|\sin(\omega t + \phi + \angle H(j\omega)). \tag{03.33}$$

We use the same metric as before for the nearness of $|H(j\omega)|$ to unity—the *dynamic error*—

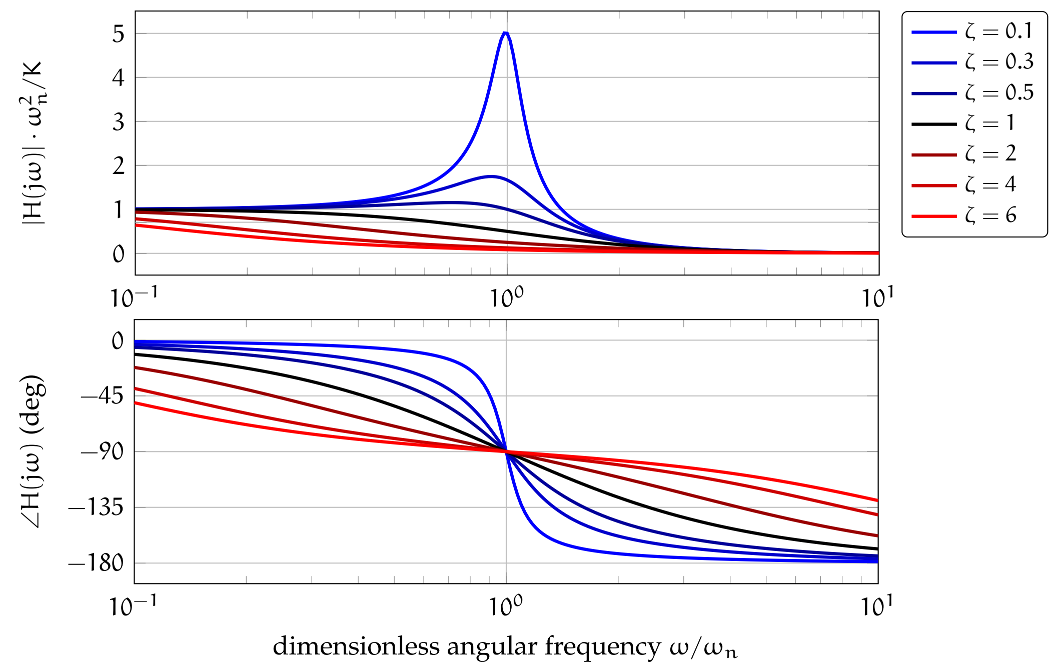

Figure 03.9: the magnitude and phase of the frequency response function $H(j\omega)$.

When $\delta(\omega) \approx 0$, the input (measurand) amplitude and output (indication) amplitude are approximately equal. Note that, according to Figure 03.9, when $\delta(\omega) \approx 0$ (i.e. $|H(j\omega)| \approx 1$), the phase lag (and therefore the time lag) is relatively small. This is ideal for second-order measurement systems.