The *steady-state* response of a linear time-invariant system to a sinusoidal input signal $u(t)$ of frequency ω is an output sinusoidal signal $y(t)$ at the same frequency as the input, but with different magnitude and phase. We determine the steady-state input/output relationship from the transfer function $G(s)$. The variation of magnitude and phase with input frequency ω is called the *Frequency Response*.

Bode Diagrams simplify the determination of the frequency response by graphing the response on logarithmic scales.

The ratio of the magnitude of the output to that of the input is called the *gain*. The magnitude portion of the Bode diagram is expressed as the

$$\text{logarithmic gain} = 20\log_{10}|G(j\omega)|$$

where the units are in *decibels* (dB).

Generally, a transfer function can be expressed in terms of factors of its poles and zeros. The advantage of the logarithmic plot is the conversion of these multiplicative factors to additive terms. Consider the general transfer function below where we substitute $s = j\omega$ to obtain the steady state sinusoidal response.

$$G(s)\,|_{s=j\omega} = G(j\omega) = \frac{Y(j\omega)}{U(j\omega)} =$$

$$= \frac{K_b \prod_{i=1}^{Q} (1 + j\omega\tau_i)}{(j\omega)^N \prod_{m=1}^{M} (j\omega\tau_m + 1) \prod_{k=1}^{R} \left[1 - (\omega/\omega_{n_k})^2 + j2\zeta_k\,(\omega/\omega_{n_k})\right]}$$

In this example, the transfer function includes Q zeros on the real axis, N poles at the origin, M poles on the real axis, and R pairs of complex conjugate poles. Interpreting the magnitude and phase of this function would be very difficult. However, if we compute *log magnitude* we have

$$\begin{aligned} 20\log|G(j\omega))| &= +20\log K_b + \\ &\quad +20\sum_{i=1}^{Q}\log|1 + j\omega\tau_i| - 20\log|(j\omega)^N| \\ &\quad -20\sum_{m=1}^{M}\log|1 + j\omega\tau_m| + \\ &\quad -20\sum_{k=1}^{R}\left|1 - (\omega/\omega_{n_k})^2 + j2\zeta_k\,(\omega/\omega_{n_k})\right| \end{aligned}$$

and the Bode diagram can be obtained by simply *adding* the plots of each term.

A separate phase angle plot is obtained as

$$\begin{aligned} \phi(\omega) &= +\sum_{i=1}^{Q}\tan^{-1}\omega\tau_i - N \times (90\,\text{deg}) \\ &\quad -\sum_{m=1}^{M}\tan^{-1}\omega\tau_m \\ &\quad -\sum_{k=1}^{R}\tan^{-1}\left[\frac{2\zeta_k\,(\omega/\omega_{n_k})}{1 - (\omega/\omega_{n_k})^2}\right] \end{aligned}$$

Therefore, there are just four different types of factors that occur in transfer functions:

1. Constant gain, K_b
2. Poles (or zeros) at the origin: $(j\omega)$
3. Poles (or zeros) on the real axis: $(j\omega\tau + 1)$
4. Complex conjugate poles (or zeros): $\left[1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2\right]$

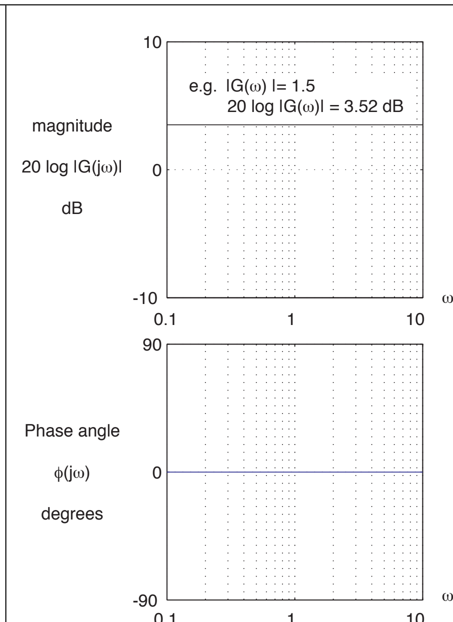
Below, the Bode Diagram magnitudes and phase angles for these four factors are described. The results can be used to obtain the Bode diagram for any transfer function.

1. Constant gain: K_b

On the Bode diagram, the log magnitude is a constant: $20\log K_b$ dB for all ω.

And, the phase angle is 0 deg for all ω.

- Since this constant term will be added to the others, increasing (or decreasing) the value of the gain simply shifts the entire log magnitude diagram up (or down) without altering its shape. The phase is unchanged.
- Notice that the Bode Diagram plots magnitude (dB) and phase (deg) on linear scales, versus frequency ω on a log scale.



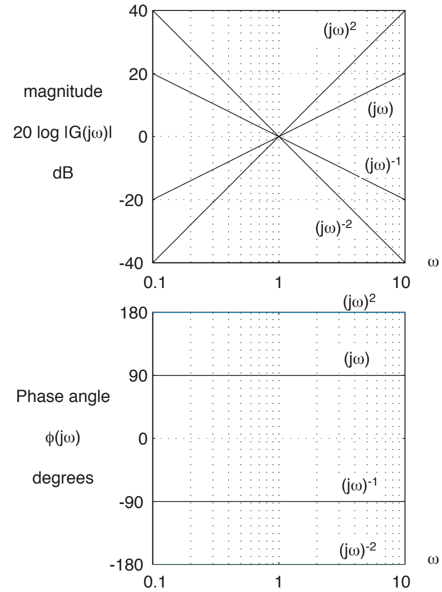
2. Poles or zeros at the origin: $(j\omega)$

For poles the log magnitude is $20\log\left|\frac{1}{(j\omega)^N}\right| = -20N\log\omega$

- This is a straight line with slope of $-20N$ dB/decade that passes through the point [0 dB, 1 r/s].
- The phase is constant with frequency: $\phi(j\omega) = -90N$ deg

For **zeros** the log magnitude is $20\log\left|(j\omega)^N\right| = +20N\log\omega$

- This is a straight line with slope of $+20N$ dB/decade that passes through the point [0 dB, 1 r/s].
- The phase is constant with frequency: $\phi(j\omega) = +90N$ deg



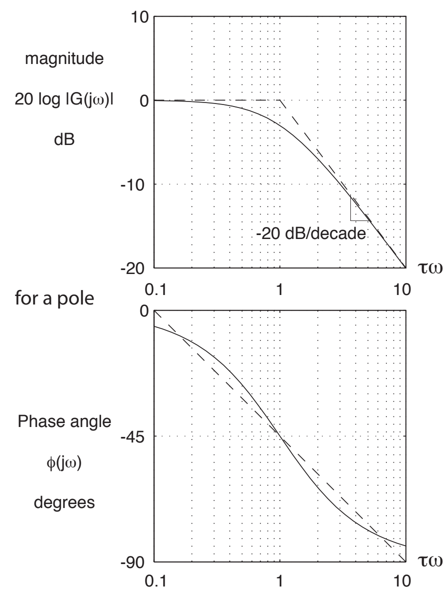
3. Poles or zeros on the real axis: $(j\omega\tau + 1)$

For poles the log magnitude is $20\log\left|\frac{1}{1+j\omega\tau}\right| = -10\log(1+\omega^2\tau^2)$

- The asymptote for $\omega \ll 1/\tau$ is a constant: $20\log 1 = 0$ dB.
- The asymptote for $\omega \gg 1/\tau$ is $-20\log\omega\tau$: a straight line with slope of -20 dB/decade. Asymptotes intersect at the break frequency $\omega_B = 1/\tau$.
- Beginning at 0 deg, the phase decreases 90 deg, and is approximate by a line with slope -45 deg /decade, that intersects -45 deg at $\omega = 1/\tau$.

For zeros the log magnitude is $20\log|1+j\omega\tau| = +10\log(1+\omega^2\tau^2)$

- The asymptote for $\omega \ll 1/\tau$ is a constant: $20\log 1 = 0$ dB.
- The asymptote for $\omega \gg 1/\tau$ is $+20\log\omega\tau$: a straight line with slope of $+20$ dB/decade. Asymptotes intersect at the break frequency $\omega_B = 1/\tau$.
- Beginning at 0 deg, the phase increases 90 deg, and is approximate by a line with slope $+45$ deg /decade, that intersects $+45$ deg at $\omega = 1/\tau$.

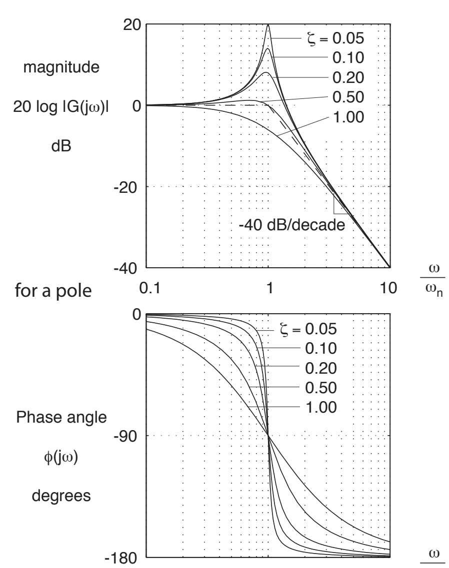


4. Complex conjugate poles (or zeros): $\left[1 + (2\zeta/\omega_n)j\omega + (j\omega/\omega_n)^2\right]$

The magnitude in decibels is

$$20\log|G(j\omega)| = -(+)10\log\left[\left(1-(\omega/\omega_n)^2\right)^2 + 4\zeta^2(\omega/\omega_n)^2\right]$$

- For $\omega \ll \omega_n$, the log magnitude is asymptotic to a straight line of constant gain: 0 dB, and the phase angle approaches 0 deg.
- For $\omega \gg \omega_n$, the log magnitude approaches $-(+)10\log(\omega/\omega_n)^4 = -(+)40\log(\omega/\omega_n)$, a straight line with slope of $-(+)40$ dB/decade. Asymptotes intersect at break frequency $\omega_B = \omega_n$.
- The resonant frequency is given by $\omega_r = \omega_n\sqrt{1-2\zeta^2}$, and the maximum magnitude is $M_{p_\omega} = |G(j\omega_r)| = 1/(2\zeta\sqrt{1-\zeta^2})$, for $\zeta < 1/\sqrt{2}$.
- Systems with very small damping are conveniently described by the resonance quality $Q = \frac{1}{2\zeta}$.



Example

Bode diagrams are conveniently plotted in MATLAB using the bode() function. However, it is important to understand how the individual factors in the transfer function combine to produce the overall magnitude and phase characteristics.

A quick way to visualize the Bode diagram is to:

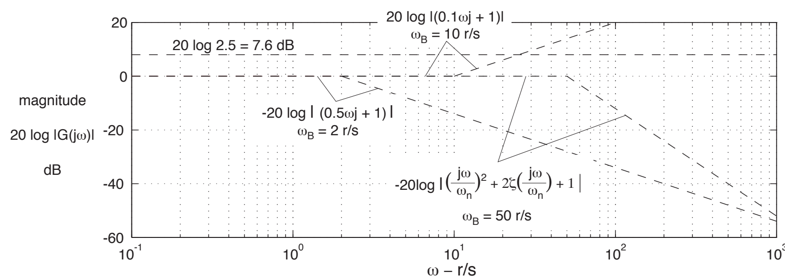
1. Rewrite the transfer function such that the real-axis and complex conjugate poles and zeros approach 1 as $s \to 0$. This normalization will simplify step 3.
2. Draw the asymptotes for each of the factors in the transfer function on the log magnitude diagram.
3. Sum the individual asymptotes to get the total asymptote.
4. Sketch the exact curve.

Suppose that the transfer function is:

$$T(s) = \frac{Y(s)}{U(s)} = \frac{1250(s+10)}{(s+2)(s^2+20s+50^2)}$$

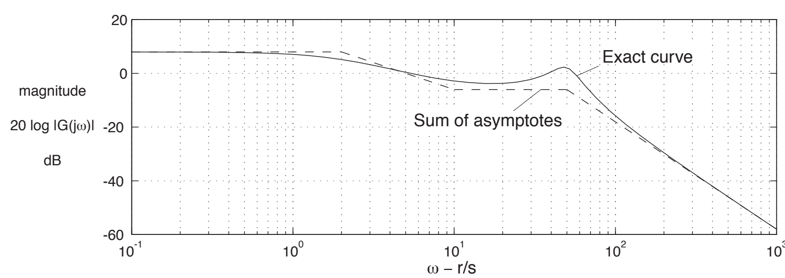
rewriting the individual factors according to step 1. we have

$$T(s) = \frac{Y(s)}{U(s)} = \frac{2.5(0.1s+1)}{(0.5s+1)\left[\left(\frac{s}{50}\right)^2 + 20\frac{s}{50^2} + 1\right]}$$



At the right, the individual and total asymptotes have been drawn at the corresponding break frequencies. Noting that the damping ratio ζ of the conjugate poles is 0.2, an approximate curve can be sketched.

The exact curve is shown, and is close to the asymptotic approximation.



Finally, the sum of the phase components corresponding to the individual factors determines the total phase.