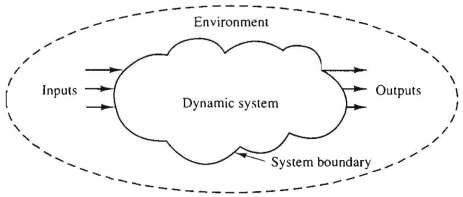
System Dynamics

I prepared these notes on system dynamics, based primarily on Rowell and Wormley's *System Dynamics: An Introduction* [1], for my own use and share it with others with the understanding that the contents are not meant to be original work of my own, but merely unofficial notes taken from Rowell and Wormley. That means the overall structure, some phrases, and most figures are copied directly from [1]. There is no guarantee that the contents are true to the book, however, so use it at your own risk. Please send any errors and typos to piconer@uw.edu.

1 Basics

1.1 State-determined systems[1, p 5]

System Dynamics can be thought of as the study of the interaction between a *system* and its *environment*. *System variables* characterize the system and can be considered *inputs* (those arising from interaction with the environment) and *outputs* (those of interest).



A *state-determined system model* is a mathematical description (*state equations*) of a system whose behavior for all $t \geq t_0$ can be determined given the initial conditions of the system at $t = t_0$ and the inputs for all $t \geq t_0$(∴ the system "forgets").

State variables are a minimum set of variables that uniquely define the system response for all t. They must be independent and complete.

Elements of systems are the primitive blocks that supply, store, and dissipate energy and together build systems. A state-determined system model is created by identifying proper elements and their interactions.

2 System elements: one-port[1, p 19]

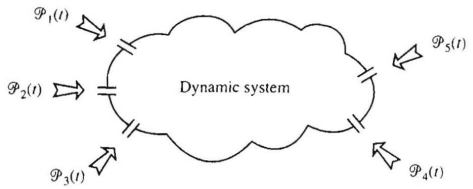
One-port elements represent the spatial locations (*ports*) where energy is transfered. These can generate, store, or dissipate energy in (but not between) the following *energy domains*: mechanical, electrical, fluid, and thermal.

2.1 First law of thermodynamics

Defining power $\mathcal{P}$ as the positive time rate of change of the total energy stored in the system $\mathcal{E}$,

$$\mathcal{P}(t) = d\mathcal{E}/dt. \tag{1}$$

We assume that this energy exchange (or *power flow*) between the system and the environment occurs through a finite number of ports.



This power flow is a result of *work* W being performed on (+) or by (−) the system and *heat* energy H flowing in (+) or out (−) of the system:

$$\mathcal{P}dt = \Delta W + \Delta H \tag{2}$$

Conservation of energy can be expressed as the sum of the n power flows into the system being equal to the time rate of change of the sum of the energy stored in the m system elements:

$$\sum_{i=1}^{n} \mathcal{P}_i(t) = \sum_{j=1}^{m} \frac{dE_j}{dt}. \tag{3}$$

This is valid for *lumped-parameter systems* comprised of *lumped-parameter elements* that represent the behavior of a region of the system that is considered to have somewhat uniform behavior (e.g. a car's velocity as opposed to the velocity of each point on the car), and can be described by ODEs. This is opposed to spatially *continuous* or *distributed* systems, which are described by PDEs.

2.2 Mechanical system elements[1, p 21]

2.2.1 Translational

Here the power flow $\mathcal{P}(t)$ is the product of the *power flow variables*: velocity $v(t)$ and collinear force $F(t)$, so

$$\mathcal{P}(t) = F(t)v(t). \tag{4}$$

Mechanisms for energy storage and dissipation are:

- *mass*: kinetic energy stored as massive translating elements,
- *spring*: potential energy stored as elastic deformation of springlike elements, and
- *damper*:energy dissipated through friction to heat.

Note that the typical linear relationships that we will see below are *ideal*, and that real springs and friction (damping) behave differently.

Ideal sources (which can give infinite power) are:

- *force source*: in which supplied force $F_s(t)$ is designated and the resulting velocity is a function of the system, and
- *velocity source*: in which supplied velocity $V_s(t)$ is designated and the resulting force is a function of the system.

2.2.2 Rotational

Here the power flow $\mathcal{P}(t)$ is the product of the *power flow variables*: angular velocity $\Omega(t)$ and torque $T(t)$ about a fixed axis, so

$$\mathcal{P}(t) = T(t)\Omega(t). \tag{5}$$

Mechanisms for energy storage and dissipation are:

- *rotational inertia*: kinetic energy stored as massive rotating elements,
- *rotational spring*: potential energy stored as elastic angular deformation of springlike elements, and
- *rotational damper*: energy dissipated through friction to heat.

Of interest here is the rotational *moment of inertia* (see [1, p 34] for J of some shapes),

$$J = \sum_{i=1}^{n} m_i r_i^2 \Rightarrow J = \int_V r^2 dm. \tag{6}$$

Ideal sources (which can give infinite power) are:

- *torque source*: in which supplied torque $T_s(t)$ is designated and the resulting angular velocity is a function of the system, and
- *angular velocity source*: in which supplied angular velocity $\Omega_s(t)$ is designated and the resulting torque is a function of the system.

2.3 Electrical system elements[1, p 37]

Here the power flow $\mathcal{P}(t)$ is the product of the *power flow variables*: current $i(t)$ and voltage drop $v(t)$, so

$$\mathcal{P}(t) = i(t)v(t). \tag{7}$$

Mechanisms for energy storage and dissipation are:

- *capacitor*: electrical energy stored as charge q in an electrostatic field,
- *inductor*: magnetic energy stored in a magnetic field, and
- *resistor*: energy dissipated through material resistivity to heat.

Ideal sources (which can produce infinite power) are:

- *current source*: in which supplied current $I_s(t)$ is designated and the resulting voltage is a function of the system, and
- *voltage source*: in which supplied voltage $V_s(t)$ is designated and the resulting current is a function of the system.

2.4 Fluid system elements[1, p 44]

Here the power flow $\mathcal{P}(t)$ is the product of the *power flow variables*: fluid volume flowrate $Q(t)$ and pressure drop $P(t)$ across the port, so

$$\mathcal{P}(t) = P(t)Q(t). \tag{8}$$

Mechanisms for energy storage and dissipation are:

- *fluid capacitor*: potential energy stored in the fluid,
- *fluid inertance*: kinetic energy stored in the fluid, and
- *fluid resistor*: energy dissipated through fluid work to heat.

Ideal sources (which can give infinite power) are:

- *flow source*: in which supplied volume flow $Q_s(t)$ is designated and the resulting pressure is a function of the system, and
- *pressure source*: in which supplied pressure $P_s(t)$ is designated and the resulting volume flow is a function of the system.

2.5 Thermal system elements[1, p 53]

Here the power flow $\mathcal{P}(t)$ is the not the product of the *power flow variables*: temperature $T(t)$ and heat flow rate $q(t)$ (the time derivative of *heat* or thermal energy H), but

$$\mathcal{P}(t) = q(t) = \frac{dH(t)}{dt}. \tag{9}$$

Mechanisms for energy storage and dissipation are:

- *thermal capacitor*: thermal energy stored as heat H, the thermal energy, and
- *resistor*: energy dissipated through transferring heat by *conduction*, *convection*, and *radiation*.

Ideal sources are:

- *heat flow source*: in which supplied heat flow $Q_s(t)$ is designated and the resulting temperature is a function of the system, and
- *temperature source*: in which supplied temperature $T_s(t)$ is designated and the resulting heat flow is a function of the system.

3 Generalized one-port elements[1, p 66]

Here we describe a generalization of the system elements of Section 2 in order to use a unified method (*linear graphs*) to analyze systems of the different energy domains.

3.1 Through- and across-variables

Through variables are continuous between the two terminals of a system element, and can only be measured by "breaking" the system and inserting a sensor (e.g. force and current).

Across variables are relative quantities because they are measured as the differences between values at each terminal of the element (e.g. velocity and voltage).

Generalized through- and *across-variables* are defined as follows.

Generalized variables

across	v
integrated across	$\mathsf{x} = \int_0^t \mathsf{v}dt + \mathsf{x}(0)$
through	f
integrated through	$\mathsf{h} = \int_0^t \mathsf{f}dt + \mathsf{h}(0)$
power passing into (non-thermal) element	$\mathcal{P}(t) = \mathsf{fv}$
work done by system on element for $0 \le t \le T$	$W = \int_0^T \mathcal{P}dt = \int_0^T \mathsf{fv}dt$

Across- and through-variables by energy domain

System	across	through	$\int$across	$\int$through
General	v	f	x	h
Translational	v	F	x	p
Rotational	Ω	T	Θ	h
Electric	v	i	λ	q
Fluid	P	Q	Λ	V
Thermal	T	q	-	H

Variable units and definitions

across	v		generalized across
	v	m/s	velocity difference
	Ω	rad/s	angular velocity difference
	v	V	voltage drop
	P	$\mathrm{N/m^2}$	pressure difference
	T	K	temperature difference
through	f		generalized through
	F	N	force
	T	N·m	torque
	i	A	current
	Q	$\mathrm{m^3/s}$	volume flow rate
	q	W	heat flow rate
$\int$across	x		generalized integrated across
	x	m	linear displacement
	Θ	rad	angular displacement
	λ	V·s	flux linkage
	Γ	$\mathrm{N \cdot s/m^2}$	pressure difference momentum
$\int$through	h		generalized integrated through
	p	$\mathrm{N \cdot s}$	momentum
	h	$\mathrm{N \cdot m \cdot s}$	angular momentum
	q	$\mathrm{A \cdot s}$	charge
	V	$\mathrm{m^3}$	volume
	H	J	heat

3.2 A-, T-, and D-type elements

3.2.1 A-type energy storage elements

A-type elements are those in which the energy stored in the element is a function of the across-variable and are called *generalized capacitances*, which, when linear, are represented by C. See [1, p 71].

A-type elements' elementary relationships

Element	Constitutive Equation	Elemental Equation	Energy
generalized	$\mathsf{h} = \mathsf{C}\,\mathsf{v}$	$\mathsf{f} = \mathsf{C}\frac{d\mathsf{v}}{dt}$	$\mathcal{E} = \frac{1}{2}\mathsf{C}\,\mathsf{v}^2$
trans. mass	$p = mv$	$F = m\frac{dv}{dt}$	$\mathcal{E} = \frac{1}{2}mv^2$
rot. inertia	$h = J\Omega$	$T = J\frac{d\Omega}{dt}$	$\mathcal{E} = \frac{1}{2}J\Omega^2$
elec. cap.	$q = Cv$	$i = C\frac{dv}{dt}$	$\mathcal{E} = \frac{1}{2}Cv^2$
fluid cap.	$V = C_f P$	$Q = C_f\frac{dP}{dt}$	$\mathcal{E} = \frac{1}{2}C_f P^2$
thermal cap.	$H = C_t T$	$q = C_t\frac{dT}{dt}$	$\mathcal{E} = C_t T$

3.2.2 T-type energy storage elements

T-type elements are those in which the energy stored in the element is a function of the through-variable and are called *generalized inductances*, which, when linear, are represented by L. See [1, p 71].

T-type elements' elementary relationships

Element	Constitutive Equation	Elemental Equation	Energy
generalized	$\mathsf{x} = \mathsf{L}\mathsf{f}$	$\mathsf{v} = \mathsf{L}\frac{d\mathsf{f}}{dt}$	$\mathcal{E} = \frac{1}{2}\mathsf{L}\,\mathsf{f}^2$
trans. spring	$x = \frac{1}{K}F$	$v = \frac{1}{K}\frac{dF}{dt}$	$\mathcal{E} = \frac{1}{2K}F^2$
tors. spring	$\Theta = \frac{1}{K_r}T$	$\Omega = \frac{1}{K_r}\frac{dT}{dt}$	$\mathcal{E} = \frac{1}{2K_r}T^2$
elec. induc.	$\lambda = Li$	$v = L\frac{di}{dt}$	$\mathcal{E} = \frac{1}{2}Li^2$
fluid inert.	$\Lambda = I_f Q$	$P = I_f\frac{dQ}{dt}$	$\mathcal{E} = \frac{1}{2}I_f Q^2$

3.2.3 D-type energy storage elements

D-type elements are those in which the energy dissipated and the element has an algebraic relationship between across and through variables and are called *generalized resistances*, which, when linear, are represented by R. See [1, p 71].

D-type elements' elementary relationships

Element	Elemental Equations		Power Dissipated
generalized	$\mathsf{f} = \frac{1}{\mathsf{R}}\mathsf{v}$	$\mathsf{v} = \mathsf{R}\mathsf{f}$	$\mathcal{P} = \frac{1}{\mathsf{R}}\mathsf{v}^2 = \mathsf{R}\,\mathsf{f}^2$
trans. damper	$F = Bv$	$v = \frac{1}{B}F$	$\mathcal{P} = Bv^2 = \frac{1}{B}F^2$
rot. damper	$T = B_r\Omega$	$\Omega = \frac{1}{B_r}T$	$\mathcal{P} = B_r\Omega^2 = \frac{1}{B_r}T^2$
elec. resist.	$i = \frac{1}{R}v$	$v = Ri$	$\mathcal{P} = \frac{1}{R}v^2 = Ri^2$
fluid resist.	$Q = \frac{1}{R_f}P$	$P = R_f Q$	$\mathcal{P} = \frac{1}{R_f}p^2 = R_f Q^2$
thermal resist.	$q = \frac{1}{R_t}T$	$T = R_t q$	none: impedes heat flow

3.3 Ideal sources

Ideal sources provide the appropriate across- or through-variable as a function of time, while the other variable depends on the system to which the source is connected. Note in the figure that the for through-variables, the arrow points in the assumed-positive direction of through-variable flow and for across-variables, the arrow points in the assumed direction of across-variable decrease or drop. [1, p 80]

(a) Through-variable source (b) Across-variable source

3.4 Causality

The elemental equations relate the through- and across-variables and must hold at all times in an element, so if either the through- or across- variable is defined, the other must be determined by the elemental equation.

If the system-defined or given variable must be differentiated in the elemental equation to obtain the other unknown variable, the element is in *derivative causality*, whereas if the system-defined or given variable must be integrated to obtain the other unknown variable, the element is in *integral causality*. All energy storage elements have either derivative or integral causality, while all dissipative elements have algebraic causality, since the elemental equations are merely algebraic.

3.5 Linearization of nonlinear elements

To approximate the behavior of a nonlinear element about an operating point, perform a Taylor-series expansion of the constitutive equation about the operating point, re-define the across- and through-variables as perturbed variables (e.g. $\mathsf{v}^* = \mathsf{v} - \mathsf{v}_0$), and through out second- and higher-order terms to obtain the perturbed constitutive equation. [1, p 84]

3.5.1 A-type element linearization

An A-type element has a single-valued monotonic constitutive relationship $\mathsf{h} = \mathcal{F}(\mathsf{v})$, which can be linearized and re-written in terms of perturbed variables $\mathsf{h}^* = \mathsf{h} - \mathsf{h}_0$ and $\mathsf{v}^* = \mathsf{v} - \mathsf{v}_0$, where h_0 and v_0 are the equilibrium values about which the linearization is being performed, as

$$\mathsf{h}^* = \mathsf{C}^*\mathsf{v}^* \tag{10a}$$

where

$$\mathsf{C}^* = \left.\frac{d\mathcal{F}(\mathsf{v})}{d\mathsf{v}}\right|_{\mathsf{v}=\mathsf{v}_0}. \tag{10b}$$

So we have a linearized elemental equation

$$\mathsf{f}^* = \mathsf{C}^*\frac{d\mathsf{v}^*}{dt}. \tag{11}$$

3.5.2 T-type element linearization

A T-type element has a single-valued monotonic constitutive relationship $\mathsf{x} = \mathcal{F}(\mathsf{f})$, which can be linearized and re-written in terms of perturbed variables $\mathsf{x}^* = \mathsf{x} - \mathsf{x}_0$ and $\mathsf{f}^* = \mathsf{f} - \mathsf{f}_0$, where x_0 and f_0 are the equilibrium values about which the linearization is being performed, as

$$\mathsf{x}^* = \mathsf{L}^*\mathsf{f}^* \tag{12a}$$

where

$$\mathsf{L}^* = \left.\frac{d\mathcal{F}(\mathsf{f})}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_0}. \tag{12b}$$

So we have a linearized elemental equation

$$\mathsf{v}^* = \mathsf{L}^*\frac{d\mathsf{f}^*}{dt}. \tag{13}$$

3.5.3 D-type element linearization

A D-type element has an algebraic relationship $\mathsf{v} = \mathcal{F}(\mathsf{f})$, which can be linearized and re-written in terms of perturbed variables $\mathsf{v}^* = \mathsf{v} - \mathsf{v}_0$ and $\mathsf{f}^* = \mathsf{f} - \mathsf{f}_0$, where v_0 and f_0 are the equilibrium values about which the linearization is being performed, as

$$\mathsf{v}^* = \mathsf{R}^*\mathsf{f}^* \tag{14a}$$

where

$$\mathsf{R}^* = \left.\frac{d\mathcal{F}(\mathsf{f})}{d\mathsf{f}}\right|_{\mathsf{f}=\mathsf{f}_0}. \tag{14b}$$

So we have a linearized elemental equation

$$\mathsf{v}^* = \mathsf{R}^*\mathsf{f}^* \tag{15}$$

4 System models: linear graphs[1, p 92]

Linear graphs are representations of lumped-parameter systems constructed from *branches*: energy ports that represent passive or source system elements, and *nodes*: points where system elements connect and define points where distinct across-variables can be measured w/respect to the system's *reference node*: a node chosen to reference other across-variables to (e.g. for mechanical systems it is typically an *inertial reference frame*).

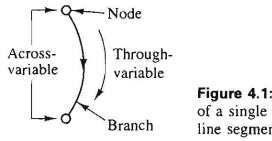
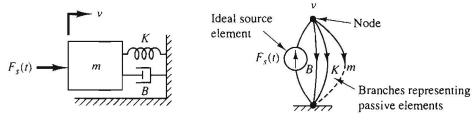


Figure 4.1: Linear graph representation of a single passive element as a directed line segment.



Note that the elements and methods of linear graphs do not assume the elements to be linear. Each branch is associated with an elemental through-variable and its across-variable is defined as the difference between the across variable at each node.

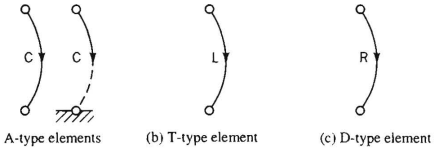
4.1 Linear graph representations of one-port elements

The generalized ideal elemental equations are

- A-type element capacitance C: $\frac{d\mathsf{v}}{dt} = \frac{1}{\mathsf{C}}\mathsf{f}$
- T-type element inductance L: $\frac{d\mathsf{f}}{dt} = \frac{1}{\mathsf{L}}\mathsf{v}$, and
- D-type element resistance R: $\mathsf{v} = \mathsf{R}\mathsf{f}$ or $\mathsf{f} = \frac{1}{\mathsf{R}}\mathsf{v}$.

The branches have the following sign convention, designated by the arrows: the arrow is drawn in the direction in which

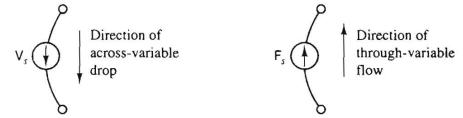
- the branch's across-variable v is assumed decreasing (in the assumed direction of the across-variable drop), **and**
- the through-variable f is defined as having a positive value.



(a) A-type elements (b) T-type element (c) D-type element

Notice that A-type elements (with the exception of electric capacitors), must have their across-variable defined with respect to a constant reference value (denoted dashed-line).

The sign conventions for sources are similar, and are shown in the following figure.



(a) Across-variable source (b) Through-variable source

4.2 Element interconnection laws

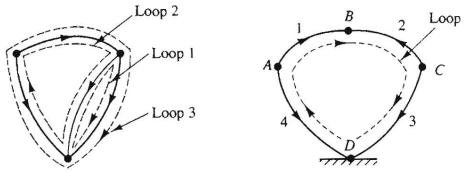
4.2.1 Compatibility

Quoting the *compatibility* law from [1, p 95]: the sum of the across-variable *drops* on the branches around any closed loop on a linear graph is identically zero, or

$$\sum_{i=1}^{N} \mathsf{v}_i = 0 \tag{16}$$

for any N elements forming a closed loop on the graph.

If a loop is drawn as in the following figures, branches with arrows pointed with the loop-arrow can be summed as positive, and those pointed against negative.



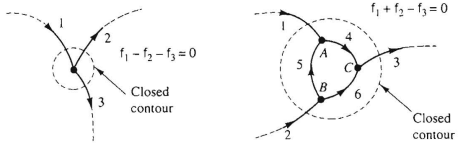
4.2.2 Continuity

Quoting the *continuity* law from [1, p 97]: the sum of through-variables flowing *into* any closed contour drawn on a linear graph is zero, or

$$\sum_{i=1}^{N} \mathsf{f}_i = 0 \tag{17}$$

for any N branches that intersect a closed contour on the graph.

If the countour is drawn as in the following figures, branches with arrow pointed *into* the closed contour are positive in the sum, those pointed out negative.



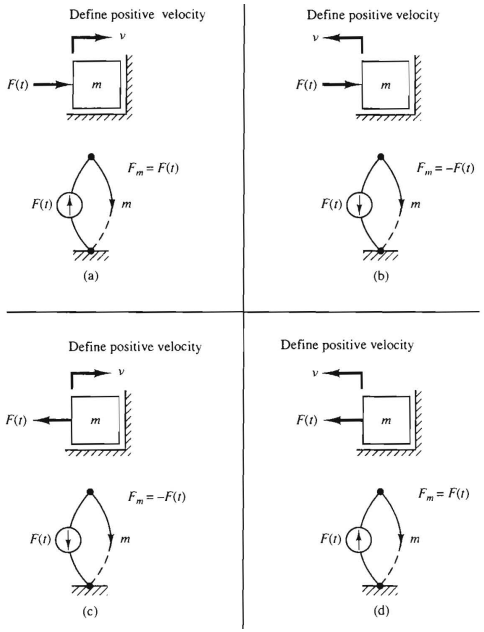
4.2.3 Series and parallel connection of elements

Analogous to circuits, with a parallel connection of branches between two points, the across-variable for each parallel branch is identical (the through-variable splits between branches). With a series connection of branches between two points, the through-variable for each series branch is identical (the across-variable varies).

4.3 Sign conventions on one-port system elements

In addition to to following the conventions from the beginning of this section (Section 4) is best to follow the following conventions:

- define the positive model source direction compatible with that of the physical system (and per beginning of this section) and
- assume arrows always point away from sources, toward grounds.



4.4 Linear graph models of systems of one-port elements

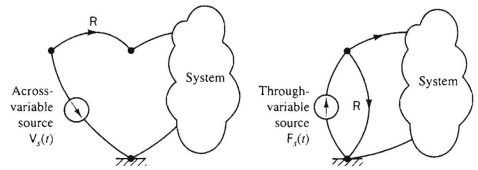
The following procedure from [1, p 101] can be used to construct *system graphs*:

- define the system boundary and analyze the physical system features to be included in the model: (a) inputs, (b) outputs of interest, (c) energy domains involved, and (d) required elements;
- draw a schematic and assign a sign convention;
- determine lumped-parameter elements: (a) source, (b) energy storage, and (c) energy dissipative;
- identify across-variables that define nodes and draw the nodes;
- determine between which nodes each element lies and draw them;
- select a sign convention for the passive elements and draw arrows (see Section 4.3); and
- select the sign conventions for source elements to be consistent with the physical model (see Section 4.3) and draw them.

4.5 Physical source modeling

Since sources are often non-ideal, the following can be useful source models:

- Thevenin equivalent (across-variable) source: an ideal across variable source V_s in series with a resistance R, which can be described by $\mathsf{v} = \mathsf{V}_s - \mathsf{Rf}$.
- Norton equivalent (through-variable) source: an ideal through variable source F_s in parallel with a resistance R, which can be described by $\mathsf{f} = \mathsf{F}_s - \frac{1}{\mathsf{R}}\mathsf{v}$.



(a) Thévenin equivalent source (b) Norton equivalent source

5 State equation formulation[1, p 120]

5.1 State variable system representation

The *state equations* are a set of differential equations with *state variables* as unknowns that completely describe the time evolution of the system, given a set of initial conditions (expressed as state variables). The system has order n, which corresponds to the number state variables (minimum set of variables that completely describe the system behavior); n is equal to the number of **independent energy storage elements** in the system.

5.1.1 State equations

The system state $\mathbf{x}(t)$ at any time t can be understood as a point in an n-dimensional *state-space* , and its time-evolution

as a trajectory on that state-space. The equation of state is

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t), \tag{18}$$

where $\mathbf{u}(t)$ is a vector of r system inputs. For a linear system, the Equation (18) becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{19}$$

where $\mathbf{A}$ is an $n \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix.

5.1.2 Output equations

System *outputs* $\mathbf{y}(t)$, an m-vector quantity, are variables of interest and can be expressed as a linear combination of state variables and inputs in the equation

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \tag{20}$$

where $\mathbf{C}$ is an $m \times n$ matrix and $\mathbf{D}$ is an $m \times r$ matrix.

5.2 Linear graph usage

If a *system graph* (oriented linear graph) is *connected*, meaning every point can be reached traveling along branches (usually works for systems of one-port elements), the procedures in this section (5) can be used.

If there are B branches in a system graph and of these S are sources, there are $2B - S$ unknowns and therefore required equations. These will be a combination of elemental, continuity, and compatibility equations.

5.2.1 Normal tree construction

A *normal tree* is constructed in order to define the system *primary variables*, *secondary variables*, system order n, state variables, continuity equations, and compatibility equations. From these, a state model can be systematically constructed by eliminating all secondary variables from the n state equations. The procedure for constructing the normal tree is thus (starting with a system graph):

1. draw the system graph nodes,
2. select tree branches using the following rules:
   (a) no loops can be created,
   (b) $N - 1$ branches must be selected (N = number of nodes), and
3. the following order:
   (a) select all across-variable sources,
   (b) select as many as possible A-type energy storage elements,
   (c) select as many as possible D-type energy dissipative elements, and
   (d) select as many as possible T-type energy storage elements.

All elements included in the normal tree are called *branches*, and others called *links*, which, when connected to the tree form loops.

If all across-variables sources cannot be included in the normal tree, they must form a loop, and compatibility is violated. If at the end of the procedure an additional branch is required, it must be a through-source, which would violate continuity since it cannot be independently specified.

All A-type elements that **cannot be** included in the normal tree and all T-type elements that **are** included in the normal tree are *dependent energy storage elements*; all others are *independent energy storage elements*, elements whose stored energy may be independently set and controlled.

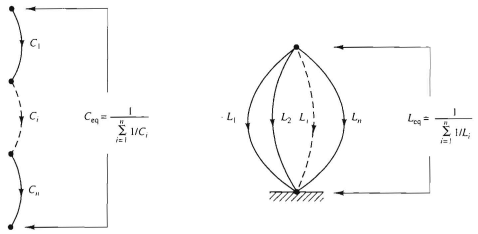
Two situations can occur when an excess of state variables may be found by the previous procedure:

1. when there are two or more A-type elements in direct **series** connection, and
2. when there are two or more T-type elements in direct **parallel** connection.

This excess can be mitigated by combining these elements as shown in the following figure with the equations

$$C_{\text{eq}} = \frac{1}{\sum_{i=1}^{n} 1/C_i} \text{ and} \tag{21a}$$

$$L_{\text{eq}} = \frac{1}{\sum_{i=1}^{n} 1/L_i}. \tag{21b}$$



5.3 State equation formulation

The following procedure can be used to give a set of state equations:

1. derive $B - S$ differential and algebraic equations in terms of primary variables only by starting with passive elemental equations and using $B - S$ compatibility and continuity equations to eliminate secondary variables (note: S(# of sources) = S_T(# of through-variable sources) + (# of across-variable sources)):
   (a) generate a normal tree (see above procedure),
   (b) identify primary variables as across-variables on tree-branches and through-variables on tree-links,
   (c) identify secondary variables as through-variables on tree-branches and across-variables on tree-links,
   (d) identify the system order n as the number of independent energy storage elements (number of A-type in normal tree plus number of T-type not in normal tree),
   (e) select the state-variables as across-variables on A-type in normal tree and through-variables on T-type in tree-links,
   (f) write $B - S$ elemental equations for passive (non-source) elements with primary variables explicitly on the left-hand-side,
   (g) write $N-1-S_A$ independent continuity equations involving only one secondary through-variable (branch through-variable) by using contours that cut only one passive branch element, and write each equation explicitly in terms of the secondary through-variable,
   (h) write $B-N+1-S_T$ independent compatibility equations involving only one secondary across-variable (link across-variable) by using loops created by replacing passive element links back into the tree, and write each equation explicitly in terms of the secondary across-variable, then
2. algebraically manipulate this set of n differential equations in the n state-variables and S specified source variables:
   (a) use the continuity and compatibility equations to eliminate all secondary variables from the elemental equations,
   (b) reduce the resulting $B - S$ equations in the primary variables to n equations in the n state-variables and S source-variables, and
   (c) write the resulting state equations in the standard form.

5.4 Systems with non-standard state equations

5.4.1 Input derivative form

When the state equations have the forms

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \mathbf{E}\dot{\mathbf{u}}, \text{ and} \tag{22}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \mathbf{F}\dot{\mathbf{u}}, \tag{23}$$

which are typically due to either (1) a compatibility equation includes the across-variable on a *dependent* A-type element and an across-variable source term or (2) a continuity equation includes the through-variable on a *dependent* T-type element and a through-variable source term.

These equations can be transformed into standard form by a change-of-state-variables $\mathbf{x}' = \mathbf{x} - \mathbf{E}\mathbf{u}$, which gives

$$\dot{\mathbf{x}}' = \mathbf{A}\mathbf{x}' + \mathbf{B}'\mathbf{u}, \text{ and} \tag{24}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}' + \mathbf{D}'\mathbf{u} + \mathbf{F}\dot{\mathbf{u}} \tag{25}$$

where $\mathbf{B}' = \mathbf{A}\mathbf{E} + \mathbf{B}$ and $\mathbf{D}' = \mathbf{C}\mathbf{E} + \mathbf{D}$.

5.5 State equation generation using linear algebra

If the combination of the elemental equations, after they have been written in terms of primary variables, is difficult, linear algebra methods can be used to obtain the state equations. See [1, p 150] for the formulae.

5.6 Nonlinear systems

The same methods for finding the state equations used above can be applied to nonlinear systems, but simplification of the elemental equations into the state equations can be trickier or impossible. See [1, p 152] for examples.

5.7 Linearization of state equations

By using Taylor series expansion in the usual way, we can linearize state equations into the form

$$\dot{\mathbf{x}}^* = \mathbf{A}\mathbf{x}^* + \mathbf{B}\mathbf{u}^* \tag{26}$$

where A and B are the Jacobian matrices whose entries are

$$a_{ij} = \left.\frac{\partial f_i(\mathbf{x},\mathbf{u})}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}, \text{ and } b_{ij} = \left.\frac{\partial f_i(\mathbf{x},\mathbf{u})}{\partial u_j}\right|_{\mathbf{x}=\mathbf{x}_0,\mathbf{u}=\mathbf{u}_0}.$$

6 Energy-transducing system elements[1, p 169]

6.1 Ideal transformers & gyrators

Whereas in the preceding, one-port elements were used to represent energy storage, dissipation, and storage in a single energy domain, we no introduce **two-port** elements called *transducers* which represent energy transfer between two energy domains (e.g. motor, rack and pinion). Two-port elements are also used in specific cases to represent energy transfer within a single energy domain (e.g. levers, gears).

Each port has a through- and across-variable associated with it in its own energy domain. We require that power ($\mathcal{P}$ = f v) flowing **into** each port to be positive:

$$\mathcal{P}_1 + \mathcal{P}_2 = 0 \tag{27}$$

$$\rightarrow \frac{v_1}{v_2} = -\frac{f_2}{f_1} = \text{TF} \tag{28}$$

$$\rightarrow \frac{v_1}{f_2} = -\frac{v_2}{f_1} = \text{GY} \tag{29}$$

In order to satisfy this requirement, the two elemental equations defined by a two-port element must be of one of the two forms that follow.

The *transformer* equations arise when across-variables relate to other across-variables and through-variables relate to other through-variables as

$$\begin{bmatrix} \text{v}_1 \\ \text{f}_1 \end{bmatrix} = \begin{bmatrix} \text{TF} & 0 \\ 0 & -1/\text{TF} \end{bmatrix} \begin{bmatrix} \text{v}_2 \\ \text{f}_2 \end{bmatrix} \tag{30}$$

where TF is called the *transformer ratio*.

The *gyrator* equations arise when across-variables relate to through-variables and through-variables relate to across-variables as

$$\begin{bmatrix} \text{v}_1 \\ \text{f}_1 \end{bmatrix} = \begin{bmatrix} 0 & \text{GY} \\ -1/\text{GY} & 0 \end{bmatrix} \begin{bmatrix} \text{v}_2 \\ \text{f}_2 \end{bmatrix} \tag{31}$$

where GY is called the *gyrator modulus*.

The following steps can be used to determine the TF (similar for GY):

1. establish positive v_1, f_1, v_2, and f_2 on a diagram,
2. determine v_1-v_2 relationship,
3. define TF as v_1/v_2, and
4. using Equation (28), define f_1-f_2 relationship

Some transformer ratio & gyrator modulus examples

	model	TF/GY	notes
transformer TF	rack & pinion	r	r: radius of pinion [1, p 176]
	gear train	$-1/N$	$N = r_1/r_2 = n_1/n_2$ (r: radii, n: teeth) [1, p 175]
	DC motor	$2NB\ell r$	see [1, p 179]
	slider-crank	$-r$	r: radius of crank [1, p 171]
	block & tackle	-2	see [1, p 172]
	lever	$-1/L$	$L = l_2/l_1$ [1, p 172]
	belt drive	R	$R = r_2/r_1$ [1, p 172]
	elec trans	$1/N$	$N = N_2/N_1$ (turns) [1, p 172]
	fluid trans	A	$A = A_2/A_1$ (area) [1, p 172]
GY	hydraulic ram	$-1/A$	A: piston area [1, p 180]
	disp pump	$-1/D$	D: vol disp/rad [1, p 171]

Ideal energy transduction models often need supplemented by passive one-port elements to take into account losses and energy storage in the energy transducer.

6.1.1 Gear ratio

The notation $\mathcal{N}$: 1 denotes that the input gear θ_1 rotates $\mathcal{N}$ times for every output gear θ_2 rotation, so $\theta_1 = -\mathcal{N}\theta_2$ or $\Omega_1 = -\mathcal{N}\Omega_2$ (note that $\mathcal{N} = 1/N$).

6.1.2 Mechanical levers

The transformer ratio TF for a lever can be found by assuming small angles and recognizing that $\theta_1 = -\theta_2$:

$$x_1 = \ell_1 \sin\theta_1 = \ell_1 \sin\theta \approx \ell_1\theta \tag{32}$$

$$x_2 = \ell_2 \sin\theta_2 = -\ell_2 \sin\theta \approx -\ell_2\theta. \tag{33}$$

Solving for $x_1 = f(x_2)$, we see $x_1 = -\frac{\ell_1}{\ell_2}x_2 = -\frac{1}{L}x_2 = \text{TF}x_2$.

6.1.3 Method for "effective" capacitances

It is often that we wish to know the "effective" capacitance (mass, inertia, etc.) of some A-type element that is connected through one or more energy-transducing element to some source element that is driving it. This is done by

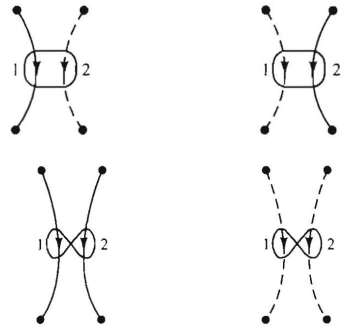
- writing down the elemental equation of the A-type element whose capacitance we would like to reflect,
- substituting into this equation compatibility, continuity, and elemental (including transducer) equations as required to transform the original equation into an A-type elemental equation that is directly connected to the source that is driving it, and
- defining the effective capacitance C_e from this elemental relationship: $\frac{dv}{dt} = \frac{1}{C_e}\text{f}$.

6.2 State-equation formulation

The process of generating a state-equation for a system including two-port elements is very close to that of Section 5.3. The main difference is in the creation of the normal tree.

6.2.1 Normal tree

Transformers (top figure) require that one or the other, but not both, branches be in the normal tree. Gyrators (bottom figure) require that both or neither branches be in the normal tree.



This requirement changes the procedure in Section 5.2.1 to the following:

1. draw the system graph nodes,
2. select tree branches using the following rules:
   (a) no loops can be created,
   (b) $N - 1$ branches must be selected (N = number of nodes),
   (c) one and only one of a transformer's two branches can be selected,
   (d) both or neither of a gyrator's two branches can be selected, and
3. the following order:
   (a) select all across-variable sources,
   (b) select as many as possible A-type energy storage elements,
   (c) select correct transducer branches, minimizing the resulting number of T-type energy storage elements in tree,
   (d) select as many as possible D-type energy dissipative elements, and
   (e) select as many as possible T-type energy storage elements.

6.2.2 State-equation generation

The state equations can be found using the normal tree creation methods of Section 6.2.1 and the procedure of Section 5.3 with the following alteration: two extra elemental equations for each transformer and gyrator will be found using Equations 30 and/or 31.

7 Operational methods for linear systems[1, p 205]

In this section we discuss *time-domain* operational mathematical and graphical methods often used to develop different system representations. The four system representations in this section are

- state equation form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u},$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u},$$

- classical form (SISO)

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1\frac{dy}{dt} + a_0 y = b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + ... + b_1\frac{du}{dt} + b_0 u, \quad (34)$$

- time-domain operator form, where H{} is the *dynamic transfer operator* for SISO systems

$$y(t) = H\{u(t)\} \quad (35)$$
$$= \frac{b_m S^m + b_{m-1}S^{m-1} + ... + b_1 S + b_0}{S^n + a_{n-1}S^{n-1} + ... + a_1 S + a_0}\{u(t)\} \quad (36)$$

  where S and S^{-1} are the differential and integral operators; or, for MIMO systems, we define the *matrix transfer operator*

$$\mathbf{y}(t) = \mathbf{H}\{\mathbf{u}(t)\}, \text{ and} \quad (37)$$

- operational block-diagram form (see [1, p 217,218,235] for vector state equation, state equation, and classical forms of block-diagrams).

7.1 Transformation from state-space equations to classical form

For first-order systems, see Section 9.1; for second-order systems, see Section 9.2.

For a quick and dirty second-order example, see [1, p 219, ex 7.5]. For higher-order or MIMO systems, a more formal method using the transfer operator is useful, and described in [1, p 228]. For higher-order SISO systems see [1, p 231-3].

These methods essentially show how to find the state equation's transfer operator $H\{\}$ and use Equation 35 to back-out an equation of the form of Equation 34.

7.2 Transformation from classical form to state-space equations (state space "realizations")

Although the state variables may not be physical, block diagrams or canonical forms may be used to transform a classical form differential equation into state-space equations. This realization (including the canonical forms, which I think easiest) is described in [1, p 233-6].

8 System properties & solution techniques[1, p 244]

8.1 System input function characterization

8.1.1 Singularity input functions

These functions are used to determine a transient (because aperiodic) system response to *discontinuous* (or are so in their derivative); they are identically zero for all $t < 0$.

1. The *unit pulse* is defined as

$$\delta_T(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1/T & \text{for } 0 < t \le T \\ 0 & \text{for } t > T \end{cases} \quad (38)$$

   where T is the pulse duration. In the limit as $T \to 0$ (or for $T << \tau$), we have the *unit impulse*,

$$\delta(t) = \begin{cases} \text{undefined} & \text{for } t = 0 \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

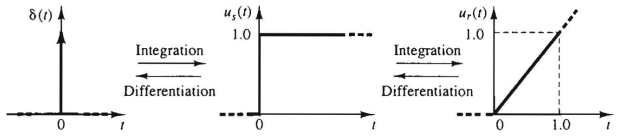
   with $\int_{-\infty}^{\infty} K\delta(t)dt = K$ where K is the *strength* of the scaled impulse.
2. The *unit step* is defined as

$$u_s(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1 & \text{for } t > 0. \end{cases} \quad (40)$$

3. The *unit ramp* is defined as

$$u_r(t) = \begin{cases} 0 & \text{for } t \le 0 \\ t & \text{for } t > 0. \end{cases} \quad (41)$$

Although not formally differentiable, these functions are related by integrals and derivatives as shown in the figure.



8.1.2 Sinusoidal input functions

These have the form $u(t) = A\sin(\omega t + \phi)$ and $u(t) = A\cos(\omega t + \phi)$, where A is the *amplitude*, ω is the *angular frequency* (rad/s), and ϕ is the *phase* (rad). The *frequency* f is found from the relationship $\omega = 2\pi f = 2\pi/T$ where T is the *period*.

8.1.3 Exponential input functions

These have the form $u(t) = e^{st}$ where s is complex in general (possibly real). When imaginary exponents are encountered, the following Euler formulas are useful:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t) \quad (42)$$
$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t) \quad (43)$$
$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t}) \quad (44)$$
$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t}) \quad (45)$$

For more properties of exponentials see [1, p 250].

8.2 Classical solution of linear differential equations

Since we can re-write any SISO state equations in classical form (see Section 7), we can use this formulation to solve for the output. Defining the *forcing function* $f(t)$ to be the right-hand-side of Equation 34, we have

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1\frac{dy}{dt} + a_0 y = f(t). \quad (46)$$

The general solution $y(t)$ to this equation is found by summing the *homogeneous solution* $y_h(t)$, found when $f(t) = 0$, and the *particular solution* $y_p(t)$, found when $f(t)$ is the specific input:

$$y(t) = y_h(t) + y_p(t). \quad (47)$$

8.2.1 Homogeneous solutions

The homogeneous solution, that of Equation 46 with $f(t) = 0$, can be found by the standard method of assuming a solution $y_h(t) = Ce^{\lambda t}$ where C is a nonzero constant, and plugging into the homogeneous equation to get the *characteristic equation*

$$\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0. \quad (48)$$

By solving this equation for $\lambda_1, \lambda_2, ..., \lambda_n$ the solution with constants $C_1, C_2, ..., C_n$. If there are n distinct roots to Equation 48,

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + ... + C_n e^{\lambda_n t}. \quad (49)$$

If there are m repeated roots to Equation 48 and $n-m$ distinct roots, then the unrepeated root solutions appear in the sum as before, but the repeated root solutions are multiplied by t^{k-1} for the k^{th} repetition (e.g. for $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \lambda = 2$, $y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda t} + C_3 t e^{\lambda t}$).

8.2.2 Particular solutions

The particular solutions, that of Equation 46 with $f(t) = f(t)$, can be found by the *method of undetermined coefficients*, which takes an educated guess based on the forcing function, checks, then chooses proper coefficients to satisfy the differential equation. The following table has common forcing function with corresponding assumed solutions.

$y_p(t)$ assumptions for method of undetermined coefficients

term in $u(t)$	assumed form for $y_p(t)$	test value
k	K_1	0
kt^n (n =1,2,...)	$K_n t^n + K_{n-1} t^{n-1} + ... + K_1 t + K_0$	0
$ke^{\lambda t}$	$K_1 e^{\lambda t}$	λ
$ke^{j\omega t}$	$K_1 e^{j\omega t}$	$j\omega$
$k\cos(\omega t)$	$K_1\cos(\omega t) + K_2\sin(\omega t)$	$j\omega$
$k\sin(\omega t)$	$K_1\cos(\omega t) + K_2\sin(\omega t)$	$j\omega$

The "test values" in the above table are to test whether or not the particular solution is a component of the homogeneous solution. If the test value is equal to any root of the characteristic equation of multiplicity m, then the assumed solution must be multiplied by t^m.

8.2.3 General (total) solutions

Equation 47 is the general solution to Equation 46. Typically, we apply the *initial conditions* (IC) after we have formed the general solution. This is not strictly required - see Section 8.3.3.

8.3 System properties

8.3.1 Stability

A system is said to be in *equillibrium* when its stave vector does not change with time; it is said to be *at rest* when it is in equillibrium and without inputs. Every system is at rest at $\mathbf{x} = \mathbf{0}$.

Assymptotic stability depends on the system response to a disturbance from an equillibrium state. If, when perturbed, it assymptotically returns to its equillibrium state, it is *assymptotically stable*. If it diverges, it is *unstable*. If it remains at its perturbed state or oscillates about the equillibrium point, it is *neutrally* or *marginally stable*.

8.3.2 Time invariance

Time-invariant systems have **A**, **B**, **C**, and **D** matrices that do not depend on time. It is defined as time-invariant if in delaying the input by T, the output is also delayed by T.

8.3.3 Superposition of LTI systems

If a system is both linear and time-invariant, it is abbreviated LTI. These systems obey the principle of *superposition*, which states:

> The response of an LTI system to a set of given ICs and an input consisting of several components,
>
> $$\mathbf{u}(t) = \mathbf{u}_1(t) + \mathbf{u}_2(t) + ... + \mathbf{u}_k(t) \qquad (50)$$
>
> may be found by determining the response to (1) the ICs with zero input and (2) each of the k individual input components and then summing all component responses to determine the total response.

One use of superposition is as a shortcut to a solution when part (either $y_{IC}(t)$ or $y_u(t)$ below) is already known. Then we can solve a forced differential equation with nonzero IC in two parts:

1. determine solution $y_{IC}(t)$ to the equation with **no inputs** and given **nonzero IC**,
2. determine solution $y_u(t)$ to the equation with given **inputs** and **zero IC**, and
3. add the two solutions for the total solution: $y(t) = y_{IC}(t) + y_u(t)$.

If the ICs are zero, then step (1) is not required; if it is unforced, then step (2) is not required; so only use this if it is both forced and has nonzero IC, and then only when one or the other is already known, or we only care about one or the other.

8.3.4 Differentiation & integration of LTI sys inputs

If $\mathbf{y}(t)$ is the output of a LTI system with input $\mathbf{u}(t)$, and the input is differentiated $d\mathbf{u}(t)/dt$ and re-applied to the system, the new output to this new input is $d\mathbf{y}(t)/dt$. (Caviot: $u(t) = 0$ for $t < 0$)

If $\mathbf{y}(t)$ is the output of a LTI system with input $\mathbf{u}(t)$, and the input is integrated $\int_0^t \mathbf{u}(t)dt$ and re-applied to the system, the new output to this new input is $\int_0^t \mathbf{y}(t)dt$. (Caviot: $u(t) = 0$ for $t < 0$)

8.4 Convolution & impulse response

See [1, p 264] for a great discussion of impulse response and convolution.

8.4.1 Impulse response

The *impulse response* $h(t)$ entirely *characterizes* a system by allowing the computation of the response to any other input $u(t)$. The impulse response is the response to the system to an impulse $\delta(t)$.

8.4.2 Convolution

Convolution is a function that maps a system's impulse response and input to the output. It is instantiated in the following equation:

$$y(t) = H\{u(t)\} \equiv u(t) \star h(t) = \int_{-\infty}^{t} u(\tau)h(t-\tau)d\tau. \qquad (51)$$

For time-limited inputs ($u(t) = 0 \;\forall\; t < t_1$ & $t_2 < t$; t_1 is often 0),

$$y(t) = \begin{cases} \int_{t_1}^{t} u(\tau)h(t-\tau)d\tau & \text{for } t < t_2 \\ \int_{t_1}^{t_2} u(\tau)h(t-\tau)d\tau & \text{for } t_2 \le t. \end{cases} \qquad (52)$$

Convolution is a linear operator and is *commutative*, *associative*, and *distributive*. This has some interesting properties for "cascaded" and "parallel" systems, see [1, p 268].

9 1st & 2nd order system response[1, p 276]

Significance of 1st&2nd order to higher order systems While most systems are of order higher than two, the rough dynamics of many systems can be approximated by a first or second order model, which are easy to derive and have useful characteristics such as time constants, natural frequencies, and damping ratios. In addition, systems of higher order than two can be considered to be comprised of first and second order components, which interact to form the higher order system. This is easily understood in terms of Bode plot construction, since real (first order) and complex conjugate pairs (second order) of poles and zeros, regardless of the order of the system, contribute similar cutoff and asymptotic characteristics.

This chapter deals with single-input systems (but typically SISO), whose state equations can be written in classical form of time-derivatives of output variables equal to time-derivatives of input variables.

The solution method used in this section is based on superposition and differentiability of LTI systems (see Section 8.3). The solutions are found by (1) finding a solution $y_{ic}(t)$: the homogeneous-IC (i.e. unforced) solution *with nonzero ICs*, (2) finding another solution $y_f(t)$: the total "forced" solution *with zero ICs*, and (3) adding solutions from (1) and (2) together:

$$y(t) = y_{ic}(t) + y_f(t) \qquad (53)$$

where we call $y_{ic}(t)$ the homogeneous-IC "unforced" response and $y_f(t)$ the *total forced response*, which we will also call the *input-output response*.

9.1 First order linear system response

The classical form of this equation is

$$\tau\frac{dy}{dt} + y(t) = f(t) \qquad (54)$$

where τ (seconds in English units for mechanical systems and in SI units in general) is the *time constant* and $f(t)$ is the forcing function. If we would like to re-write a first-order state equation in this form, we can easily do so by the equation

$$-\frac{1}{a}\frac{dy}{dt} + y(t) = -\frac{d}{a}\frac{du}{dt} + \frac{ad-bc}{a}u(t) \qquad (55)$$

where a, b, c, and d are the scalar state "matrices".

9.1.1 The homogeneous-IC (unforced) response $y_{ic}(t)$

This is found by find the homogeneous response to Equation (54) and applying the initial condition $y_{ic}(0) = y(0)$ to obtain

$$y_{ic}(t) = y(0)e^{-t/\tau}. \qquad (56)$$

9.1.2 The characteristic response $y_u(t)$

The characteristic response is *not just a particular solution*, but the general solution with zero ICs. The equation is

$$y_u(t) = y_h(t) + y_p(t) \tag{57}$$
$$= Ce^{-t/\tau} + y_p(t) \tag{58}$$

where C is found by the IC $y_u(0) = 0$ and $y_p(t)$ is the particular solution for the input $u(t)$ (note: not the forcing function $f(t)$). *Input-output responses* describe the response of a system to inputs of the form (all non-stochastic inputs) $f(t) = q_1\dot{u} + q_0 u$. The following table has characteristic and input-output (total forced) responses for the singularity inputs.

Some Responses of System $\tau\dot{y} + y = q_1\dot{u} + q_0 u$

$u(t)$	Characteristic response $y_u(t)$	Input-output (total forced) response $y_f(t)$ for $t \geq 0$
$\delta(t)$	$\frac{1}{\tau}e^{-t/\tau}$	$\frac{q_1}{\tau}\delta(t) + \left(\frac{q_0}{\tau} - \frac{q_1}{\tau^2}\right)e^{-t/\tau}$
$u_s(t)$	$1 - e^{-t/\tau}$	$q_0 - \left(q_0 - \frac{q_1}{\tau}\right)e^{-t/\tau}$
$u_r(t)$	$t - \tau(1 - e^{-t/\tau})$	$q_0 t + (q_1 - q_0\tau)(1 - e^{-t/\tau})$

9.2 Second order linear system response

The classical form of this equation is

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = f(t) \tag{59}$$

where ω_n is the *undamped natural frequency*, ζ is the (dimensionless) *damping ratio*, and $f(t)$ is a forcing function.

9.2.1 Transforming from state equations to classical form

If we would like to re-write state equations in this classical form, we must decide which output we desire. For an output of one or the other state variable, we can use Cramer's rule

$$x_1 = \frac{\det\begin{bmatrix} b_1 & -a_{12} \\ b_2 & S - a_{22} \end{bmatrix}}{\det\left[S\mathbf{I} - \mathbf{A}\right]}u \tag{60}$$

$$x_2 = \frac{\det\begin{bmatrix} S - a_{11} & b_1 \\ -a_{21} & b_2 \end{bmatrix}}{\det\left[S\mathbf{I} - \mathbf{A}\right]}u \tag{61}$$

then re-arrange. The denominator $\det\left[S\mathbf{I} - \mathbf{A}\right]$ is very important, since it determines the system response of every variable in a second-order system. Essentially, it describes the dynamics of the system without inputs (since it creates the LHS of the classical equation (59)), so we have the equality

$$\det\left[S\mathbf{I} - \mathbf{A}\right] = \frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t). \tag{62}$$

So, with only the $\mathbf{A}$ matrix, we can determine the system properties

$$\omega_n = \sqrt{a_{11}a_{22} - a_{12}a_{21}} \tag{63}$$
$$\zeta = -\frac{1}{2\omega_n}(a_{11} + a_{22}) = \frac{-(a_{11} + a_{22})}{2\sqrt{a_{11}a_{22} - a_{12}a_{21}}}. \tag{64}$$

To assign to the output y a non-state-variable, see [1, p 298, Eqs 9.54,55]. Of course, ω_n and ζ, and therefore the LHS of Equation (59), are *unchanged for any choice of output y*.

If we would like to develop homogeneous-IC solutions, we must know ICs, typically output ICs $y(0)$ and $\dot{y}(0)$. If we have state equations, the output ICs can be found from the state and output equations to be

$$y(0) = c_1x_1(0) + c_2x_2(0) \tag{65}$$
$$\dot{y}(0) = c_1\dot{x_1}(0) + c_2\dot{x_2}(0) = c_1(a_{11}x_1(0) + a_{12}x_2(0)) + c_2(a_{21}x_1(0) + a_{22}x_2(0)). \tag{66}$$

9.2.2 The homogeneous-IC (unforced) response $y_{ic}(t)$

The assumed homogeneous solution

$$y_h(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} \tag{67}$$

can be found by the characteristic equation

$$\det\left[\lambda\mathbf{I} - \mathbf{A}\right] = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{68}$$

to be

$$\lambda_1,\, \lambda_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{69}$$

In the following list, homogeneous-IC solutions $y_{ic}(t)$ will be developed for different situations arising from Equation (69). Note that in the following, it is always assumed that $y_{ic}(0) = y_0$ and $\dot{y_{ic}}(0) = 0$!

- ***$y_{ic}(t)$ for overdamped system: $\zeta > 1$***
  The roots of the characteristic equation are

$$\lambda_1,\, \lambda_2 = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \tag{70}$$

and with $\zeta > 1$, we will get real-valued λ_1 and λ_2. So from Equation (67) and the ICs $y_{ic}(0) = y_0$ and $\dot{y_{ic}}(0) = 0$, we have

$$y_{ic}(t) = y_0\left(\frac{-\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}}e^{-\zeta - \sqrt{\zeta^2 - 1}\omega_n t} - \frac{-\zeta - \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}}e^{-\zeta + \sqrt{\zeta^2 - 1}\omega_n t}\right) \tag{71}$$

which is the sum of two decaying exponentials with

$$\tau_1 = -\frac{1}{\lambda_1} \qquad \text{and} \qquad \tau_2 = -\frac{1}{\lambda_2}. \tag{72}$$

This response has neither *overshoot* nor *oscillation*.

- ***$y_{ic}(t)$ for critically-damped system: $\zeta = 1$***
  The roots of the characteristic equation are equal:

$$\lambda_1 = \lambda_2 = -\omega_n \tag{73}$$

So from Equation (67), realizing we must include a factor of t, and the ICs $y_{ic}(0) = y_0$ and $\dot{y_{ic}}(0) = 0$, we have

$$y_{ic}(t) = y_0\left(e^{-\omega_n t} + \omega_n te^{-\omega_n t}\right). \tag{74}$$

- ***$y_{ic}(t)$ for underdamped system: $0 \leq \zeta < 1$***
  The roots of the characteristic equation are

$$\lambda_1,\, \lambda_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -\zeta\omega_n \pm j\omega_d. \tag{75}$$

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ is the *damped natural frequency*. So from Equation (67), using Euler's formulas to write in terms of trigonometric functions, and the ICs $y_{ic}(0) = y_0$ and $\dot{y_{ic}}(0) = 0$, we have

$$y_{ic}(t) = y_0\frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}\cos(\omega_d t - \psi) \tag{76}$$

where the phase angle

$$\psi = \tan^{-1}\frac{\zeta}{\sqrt{1 - \zeta^2}}. \tag{77}$$

Note that as ζ *increases*, ω_d *decreases*. It is sometimes useful to know the amplitude Decay Ratio (DR) for $y_{ic}(t) \neq 0$:

$$\text{DR} = \frac{y_{ic}(t + T_p)}{y_{ic}(t)} = e^{-2\pi\zeta/\sqrt{1-\zeta^2}} \tag{78}$$

where DR is the ratio of amplitude at time $(t + T_p)$ and time t, where $T_p = 2\pi/\omega_d$.

- ***$y_{ic}(t)$ for unstable system: $\zeta < 0$***
  The roots of the characteristic equation have positive real parts, and result in an exponentially increasing, unstable response. If $-1 \leq \zeta \leq 0$, the response will oscillate with increasing amplitude; if $\zeta < -1$, the amplitude will grow exponentially.

9.2.3 The characteristic response $y_u(t)$

The input-output (total forced) response $y_f(t)$ is the general solution to the equation

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = q_2\frac{d^2u(t)}{dt^2} + q_1\frac{du(t)}{dt} + q_0u(t) \tag{79}$$

with ICs $y(0) = \dot{y}(0) = 0$. State in another way,

$$y_f(t) = y_h(t) + y_p(t) \tag{80}$$
$$= C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + y_p(t) \tag{81}$$

where C_1 and C_2 are found by applying the ICs $y(0) = \dot{y}(0) = 0$, and $y_p(t)$ is found by the method of undetermined coefficients (see Section 8.2.2).

Note that the RHS of Equation (79), the forcing function $f(t)$ contains a *linear* combination of the input $u(t)$ and its time derivatives. From superposition and derivative/integral properties of LTI systems, we can find the input-output (total forced) response $y_f(t)$ by finding the characteristic response $y_u(t)$ to the input $u(t)$ and differentiating, multiplying, and summing to get the general (total) response (see Section 9.2.4 for more details).

Since we can combine responses in this way, we need only find the characteristic response $y_u(t)$ for $u(t)$, then we can find the total forced response $y_f(t)$ for any linear combination of $u(t)$ that may arise in the RHS of (79), $f(t)$. The characteristic response $y_u(t)$ is shown in the table below for the singularity inputs.

Responses of System $\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2y(t) = g(t)$

Damping ratio	$g(t)$	Characteristic response $y_u(t)$
$0 \le \zeta < 1$	$\delta(t)$	$\frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t)$
	$u_s(t)$	$\frac{1}{\omega_n^2}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \psi)\right)$
	$u_r(t)$	$\frac{1}{\omega_n^2}\left(t + \frac{e^{-\zeta\omega_n t}}{\omega_n}\left(2\zeta\cos\omega_d t + \frac{2\zeta^2-1}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) - \frac{2\zeta}{\omega_n}\right)$
$\zeta = 1$	$\delta(t)$	$te^{-\omega_n t}$
	$u_s(t)$	$\frac{1}{\omega_n^2}\left(1 - e^{-\omega_n t} - \omega_n te^{-\omega_n t}\right)$
	$u_r(t)$	$\frac{1}{\omega_n^2}\left(t - \frac{2}{\omega_n}e^{-\omega_n t} + te^{-\omega_n t} - \frac{2}{\omega_n}\right)$
$\zeta > 1$	$\delta(t)$	$\frac{1}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right)$
	$u_s(t)$	$\frac{1}{\omega_n^2}\left(1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\tau_1e^{-t/\tau_1} - \tau_2e^{-t/\tau_2}\right)\right)$
	$u_r(t)$	$\frac{1}{\omega_n^2}\left(t - \frac{2\zeta}{\omega_n} + \frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\tau_1^2e^{-t/\tau_1} - \tau_2^2e^{-t/\tau_2}\right)\right)$

Note: in the above table $\tau_1 = -1/\lambda_1$ and $\tau_2 = 1/\lambda_2$ as in Equations (72).

9.2.4 The general (total) response $y(t)$

As mentioned above, we can find the total response to an input $u(t)$ by using the principles of superposition and differentiability/integrability of LTI systems. The input $u(t)$ always enters the equation in the form

$$f(t) = q_2\frac{d^2u(t)}{dt^2} + q_1\frac{du(t)}{dt} + q_0u(t). \tag{82}$$

If we find the response of the system to the forcing function $g(t) = u(t)$, $y_u(t)$, from the above table, then we can construct the total forced response to $g(t) = f(t)$, $y_f(t)$, by the equation

$$y_f(t) = q_2\frac{d^2y_u(t)}{dt^2} + q_1\frac{dy_u(t)}{dt} + q_0y_u(t). \tag{83}$$

If $q_1 = q_2 = 0$, the total forced response is just those solutions in the table scaled by q_0.

The general (total) response $y(t)$ to an input $u(t)$ entering through a forcing function $f(t)$ with ICs $y(0) = y_0$ and $\dot{y}(0) = 0$ is the sum of the homogeneous-IC response $y_{ic}(t)$ and the total forced response $y_f(t)$:

$$y(t) = y_{ic}(t) + y_f(t). \tag{84}$$

10 General solution of the linear state equations[1, p 331]

10.1 State variable response of linear systems

10.1.1 Homogeneous state response

The homogeneous state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{85}$$

with an arbitrary set of ICs $\mathbf{x}(0)$.

The assumed solution is

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0) \tag{86}$$

where $\mathbf{\Phi}(t) = e^{\mathbf{A}t}$ is the *state transition matrix* in terms of a *matrix exponential*. For more info about state transition matrices (including how to compute them) and matrix exponentials, see Section 10.3.

10.1.2 Forced state response

The forced state equation with input $\mathbf{u}(t)$ is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}. \tag{87}$$

with an arbitrary set of ICs $\mathbf{x}(0)$.

The solution is typically written in one of the following forms

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{88}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \tag{89}$$

where τ is a dummy integration variable. Note that in Equation (89) the integral is the convolution integral.

10.2 System output response

The output equation is

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \tag{90}$$

the homogeneous response is

$$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{x}(0), \tag{91}$$

and the forced response is

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t). \tag{92}$$

10.3 State transition matrix $\mathbf{\Phi}(t)$

10.3.1 Properties of $\mathbf{\Phi}(t)$ and $e^{\mathbf{A}t}$

The state transition matrix is defined as

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} \tag{93}$$

where the matrix exponential $e^{\mathbf{A}t}$ is defined in the table below as power series. Although this series always converges, rarely does so quickly, so other methods (see Section **??**) typically are used.

Here are some properties of the state transition matrix:

1. $\mathbf{\Phi}(0) = \mathbf{I}$,
2. $\mathbf{\Phi}(-t) = \mathbf{\Phi}^{-1}(t)$ which gives $\mathbf{x}(-t) = \mathbf{\Phi}^{-1}(t)\mathbf{x}(0)$,
3. $\mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2) = \mathbf{\Phi}(t_1) + \mathbf{\Phi}(t_2)$ which gives $\mathbf{x}(0) = \mathbf{\Phi}(-t_0)\mathbf{x}(t_0)$ and $\mathbf{x}_h(t) = \mathbf{\Phi}(t - t_0)\mathbf{x}(t_0)$, and
4. if $\mathbf{A}$ is a diagonal matrix, $e^{\mathbf{A}t}$ is a diagonal matrix with the diagonal elements $e^{a_{ii}t}$.
5. $\mathbf{x}(nT) = [\mathbf{\Phi}(T)]^n\,\mathbf{x}(0)$ $\quad \forall\ n = 1, 2, \ldots$ so if we know $\mathbf{\Phi}(t)$ at some time T, we can easily find it for integer multiples of T.

Some matrix exponential properties

Description	Matrix exponential property
definition	$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \ldots$
when $t = 0$	$e^{\mathbf{A}0} = \mathbf{I}$
inverse	$e^{-\mathbf{A}t} = \left(e^{\mathbf{A}t}\right)^{-1}$
shifting in t	$e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2}$
-	$e^{(\mathbf{A}_1+\mathbf{A}_2)t} = e^{\mathbf{A}_1t}e^{\mathbf{A}_2t}$ only if $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$
derivative	$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$
integral	$\int_0^t e^{\mathbf{A}t}dt = \mathbf{A}^{-1}\left(e^{\mathbf{A}t} - \mathbf{I}\right) = \left(e^{\mathbf{A}t} - \mathbf{I}\right)\mathbf{A}^{-1}$ if $\mathbf{A}^{-1}$ exists; otherwise defined by the series

10.3.2 System Eigenvalues and Eigenvectors

We are finding homogeneous solutions of the form

$$x_i(t) = \sum_{j=1}^{n} m_{ij}e^{\lambda_j t} \tag{94}$$

where m_{ij} are constant coefficients that depend on the system structure and initial conditions $\mathbf{x}(0)$, i.e.

$$\mathbf{x}_h(t) = \mathbf{M}\begin{bmatrix} e^{\lambda_1 t} \\ e^{\lambda_1 t} \\ \vdots \\ e^{\lambda_n t} \end{bmatrix} \tag{95}$$

where

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ m_{21} & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \ldots & m_{nn} \end{bmatrix}. \tag{96}$$

Defining $\mathbf{M}$ in terms of column vectors,

$$\mathbf{M} = \left[\mathbf{m}_1 \mid \mathbf{m}_2 \mid \ldots \mid \mathbf{m}_n\right], \tag{97}$$

if we plug Equation (95) into Equation (85), we get

$$\lambda_i\mathbf{m}_i = \mathbf{A}\mathbf{m}_i \qquad i = 1, 2, \ldots, n \tag{98}$$

which is the *eigenvalue/eigenvector* problem. The homogeneous response is determined by the eigenvalues and eigenvectors of $\mathbf{A}$. Equation (98) can be re-written as

$$[\lambda_i\mathbf{I} - \mathbf{A}]\mathbf{m}_i = \mathbf{0}, \tag{99}$$

for which a non-trivial solution requires

$$\delta(\lambda_i) = \det[\lambda_i\mathbf{I} - \mathbf{A}] = 0 \tag{100}$$

which is defined as the *characteristic equation* of $\mathbf{A}$.

> *Eigenvalues* are the n roots of the characteristic equation, λ_i.

For a physical system, the eigenvalues are either real or occur in complex conjugate pairs.

For each eigenvalue λ_i, there is an *eigenvector* $\mathbf{m}_i$ found by substituting into Equation (100). No unique solution exists, since $\mathbf{m}_i = \alpha\mathbf{m}_i \ \ \forall\ m \neq 0$.

The *modal matrix* $\mathbf{M}$ is defined as the matrix made of an arbitrary set of corresponding eigenvectors $\mathbf{m}_i$,

$$\mathbf{M} = \left[\mathbf{m}_1 \mid \mathbf{m}_2 \mid \ldots \mid \mathbf{m}_n\right] \tag{101}$$

Now we can write Equation (95) as

$$\mathbf{x}_h(t) = \mathbf{M}e^{\mathbf{\Lambda}t}\boldsymbol{\alpha} \tag{102}$$

$$= \alpha_1\mathbf{m}_1e^{\lambda_1 t} + \alpha_2\mathbf{m}_2e^{\lambda_2 t} + \ldots + \alpha_n\mathbf{m}_ne^{\lambda_n t} \tag{103}$$

where

$$e^{\mathbf{\Lambda}t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{bmatrix} \tag{104}$$

and (103) is a very useful form, showing that the solution is a linear combination of eigenvalue exponentials, in the directions of the eigenvectors, weighted by constants determined by the initial conditions. *For distinct eigenvalues*

$$\boldsymbol{\alpha} = \mathbf{M}^{-1}\mathbf{x}(0). \tag{105}$$

Note that the eigenvector columns of $\mathbf{M}$ must match the columns in which the corresponding eigenvalue appears in $e^{\mathbf{\Lambda}t}$. This gives the state transition matrix *for n distinct eigenvalues* (since $\mathbf{M}^{-1}$ does not exist otherwise)

$$\boxed{\mathbf{\Phi}(t) = \mathbf{M}e^{\mathbf{\Lambda}t}\mathbf{M}^{-1}}. \tag{106}$$

This is itself a convenient way of computing $\mathbf{\Phi}(t)$ (note: the Voyage 200 calculator can compute eigenvalues and eigenvectors) and leads to the important result:

> the homogeneous response of any state variable in the system from any ICs $\mathbf{x}(0)$ is a linear combination of n modal components $e^{\lambda_i t}$ where λ_i are the eigenvalues of A.

10.3.3 Systems with complex eigenvalues

If some roots of the characteristic equation arise in complex conjugate pairs $\lambda_{i,i+1} = \sigma \pm j\omega$, the modal matrix will have corresponding terms $e^{(\sigma\pm j\omega)t} = e^{\sigma t}e^{\pm j\omega}$, which can be converted into trigonometric form using the Euler relationships (44) and (45) to $e^{\sigma t}\sin\omega t$ or $e^{\sigma t}\cos\omega t$.

10.3.4 Systems with repeated eigenvalues

These systems are not dealt with in [1], but a preliminary discussion is presented here in Sec 10.3.9. The methods developed in 10.3.6 for finding the response do not apply to these systems.

10.3.5 Stability of linear systems

The definition of *asymptotic stability* is equivalent to stating that the homogeneous response of all state variables must decay to zero in the absence of an input:

$$\lim_{t\to\infty} x_i(t) = 0 \qquad \forall\ i = 1, 2, \ldots, n. \tag{107}$$

This leads to the summary:

> A linear system described by state equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ is asymptotically stable iff all eigenvalues of the matrix $\mathbf{A}$ have negative real parts.

Asymptotic instability occurs when at least one eigenvalue has a positive real part. *Marginal asymptotic stability* occurs when at least one eigenvalue has zero real part; if it also has a zero imaginary part the system will get "stuck" at some constant nonzero state, if instead it also has a nonzero imaginary part the system will oscillate with constant amplitude about a finite state.

10.3.6 Transformation of state variables & modal decomposition

For general linear transformations of the state variables

$$\mathbf{x} = \mathbf{P}\mathbf{q} \tag{108}$$

see [1, p 349] for a full discussion.

Using these methods, if a system has n distinct eigenvalues, we can transform the state variable s.t. $\mathbf{A}$ becomes a diagonal matrix which can provide insight into the internal structure of a system. The transformed state equations are

$$\dot{\mathbf{q}} = \mathbf{\Lambda}\mathbf{q} + \mathbf{B}'\mathbf{u} \tag{109}$$

$$\mathbf{y} = \mathbf{C}'\mathbf{q} + \mathbf{D}\mathbf{u} \tag{110}$$

where $\mathbf{q} = \mathbf{M}^{-1}\mathbf{x}$ and $\mathbf{M}$ is the modal matrix of eigenvectors,

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}, \tag{111}$$

$\mathbf{B}' = \mathbf{M}^{-1}\mathbf{B}$, and $\mathbf{C}' = \mathbf{C}\mathbf{M}$.

This gives $\mathbf{\Phi}(t) = e^{\mathbf{\Lambda}t}$ as given by Equation 104 (the modal matrix of eigenvectors for the decoupled system is simply the identity matrix). This gives the homogeneous response in the simple uncoupled form

$$q_i(t) = q_i(0)e^{\lambda_i t}. \tag{112}$$

Furthermore, in this form the solution can be written more directly as a modal decomposition

$$\mathbf{x}(t) = q_1(t)\mathbf{m}_1 + q_2(t)\mathbf{m}_2 + \dots + q_n(t)\mathbf{m}_n \tag{113}$$

where $q_i(t)$ is the i-th component of $\mathbf{q}$. This equation also applies when the modal matrix $\mathbf{M}$ comprised of both eigenvectors and generalized eigenvectors [2, p 312], so it is most certainly true that any solution to the state equation can be written in terms of its modal components. This is obvious when we think geometrically: the set of eigenvectors and generalized eigenvectors, since by definition linearly independent, constitutes a basis for the state space Σ. The fact that the input is mapped by $\mathbf{B}$ to Σ also makes the next point apparent: even the response of a system to an input may be written in terms of its modal components, and (113) still applies:

$$\mathbf{B}(t)\mathbf{u}(t) = \beta_1(t)\mathbf{m}_1 + \beta_2(t)\mathbf{m}_2 + \dots + \beta_n(t)\mathbf{m}_n. \tag{114}$$

The state equation can now be written as

$$\begin{aligned} \dot{q_1}(t)\mathbf{m}_1 + \dot{q_2}(t)\mathbf{m}_2 + \dots + \dot{q_n}(t)\mathbf{m}_n = \\ q_1(t)\mathbf{A}\mathbf{m}_1 + q_2(t)\mathbf{A}\mathbf{m}_2 + \dots + q_n(t)\mathbf{A}\mathbf{m}_n \\ +\beta_1(t)\mathbf{A}\mathbf{m}_1 + \beta_2(t)\mathbf{A}\mathbf{m}_2 + \dots + \beta_n(t)\mathbf{A}\mathbf{m}_n. \end{aligned}$$

For the special case of n eigenvectors, the state equation becomes

$$\begin{aligned} (\dot{q_1}(t) - \lambda_1 q_1(t) - \beta_1)\mathbf{m}_1 + (\dot{q_2}(t) - \lambda_2 q_2(t) - \beta_2)\mathbf{m}_2 + \dots \\ + (\dot{q_n}(t) - \lambda_n q_n'(t) - \beta_n)\mathbf{m}_n = \mathbf{0}. \end{aligned}$$

Since the set of eigenvectors $\{\mathbf{m}_i\}$ is linearly independent,

$$\dot{q_n}(t) = \lambda_n q_n(t) + \beta_n \qquad \text{for } i = 1, 2, \dots, n. \tag{115}$$

So for the case of a constant $\mathbf{A}$ matrix with a full set of eigenvectors, the system is completely described by a set of n uncoupled scalar equations whose solutions are of the form

$$q_i(t) = e^{(t-t_0)\lambda_i} q_i(t_0) + \int_{t_0}^{t} e^{(t-\tau)\lambda_i} \beta_i(\tau) d\tau. \tag{116}$$

For systems without a full set of eigenvectors, a similar mapping can be performed using the modal matrix $\mathbf{M}$ comprised of both eigenvectors and generalized eigenvectors [2, p 312]. This transformation into *normal form* results in a system

$$\dot{\mathbf{q}} = \mathbf{J}\mathbf{q} + \mathbf{B}'\mathbf{u} \tag{117}$$

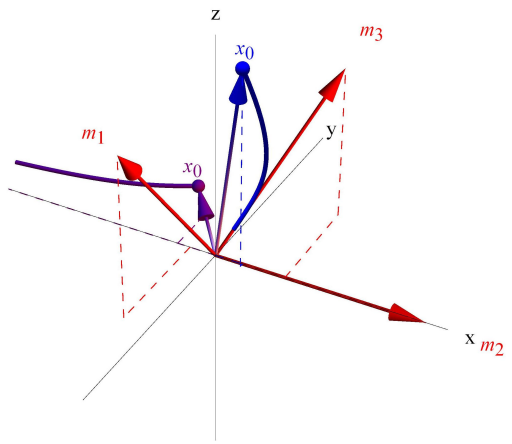
$$\mathbf{y} = \mathbf{C}'\mathbf{q} + \mathbf{D}\mathbf{u} \tag{118}$$

where $\mathbf{q} = \mathbf{M}^{-1}\mathbf{x}$, $\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ is the Jordan canonical form matrix (see Sec 10.3.9), $\mathbf{B}' = \mathbf{M}^{-1}\mathbf{B}$, and $\mathbf{C}' = \mathbf{C}\mathbf{M}$. Note that this system is as nearly decoupled as possible, but not fully so.

10.3.7 The invariant subspace

The *invariant subspace* of a linear map $\mathbf{P} : V \to V$ for a vector space V is a subspace $\nu \subseteq V$ for which $\mathbf{P}(\nu)$ is contained in ν. This subspace can be termed $\boldsymbol{P}$*-invariant*. The space spanned by eigenvectors of $\mathbf{P}$ is $\mathbf{P}$-invariant. A subspace spanned by the eigenvectors (assuming $\mathbf{P}$ has n distinct eigenvectors) will be mapped by $\mathbf{P}$ back to the same subspace.

This is important in linear systems theory when we are interested in exciting specific modes of the system without exciting others. To excite a specific mode or modes (which span an invariant subspace) with only initial conditions, choose an initial condition $\mathbf{x}(0)$ that lies in the desired invariant subspace, i.e. select $\mathbf{x}(0)$ to be a linear combination of the eigenvectors spanning the invariant subspace.



The case of real eigenvectors is easiest. We desire an initial condition that lies in the invariant subspace of $\mathbf{A}$, and this can by achieved by selecting the coefficients of the linear combination of eigenvectors (which happen to be the initial conditions of the diagonalized system),

$$\mathbf{x}(0) = \mathbf{M}\boldsymbol{\alpha} \tag{119}$$

$$= \alpha_1\mathbf{m}_1 + \alpha_2\mathbf{m}_2 + \dots + \alpha_n\mathbf{m}_n \tag{120}$$

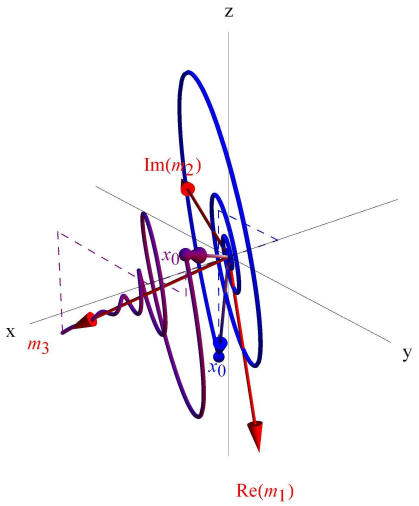
$$= \mathbf{M}\mathbf{q}(0). \tag{121}$$

So select α_i of only the desired modes to be nonzero, and the resulting linear combination yields an $\mathbf{A}$-invariant initial condition $\mathbf{x}(0)$. The above figure shows two examples of initial conditions for such a system: in blue, one selected by the preceding method and lying in the invariant plane spanned by the eigenvectors of the desired modes (1 and 3); in purple, one selected that did not lie in the that plane and required the excitation of the second mode, which in this case was unstable (probably why we didn't want to wake it up).

The case of complex eigenvectors is a bit more subtle. The real $\mathbf{A}$-invariant subspace spanned by a complex conjugate pair of eigenvectors $\mathbf{m}_1$ and $\mathbf{m}_2$ is what we are concerned with, since our initial conditions must be real. This real subspace is spanned by

$$\{\mathrm{Re}[\mathbf{m}_1], \mathrm{Im}[\mathbf{m}_1]\} \text{ or } \{\mathrm{Re}[\mathbf{m}_2], \mathrm{Im}[\mathbf{m}_2]\}. \tag{122}$$

This arises from recognizing the requirement that, in order to obtain a real initial condition, the linear combination coefficient vector $\boldsymbol{\alpha}$ must contain complex conjugate pairs of complex numbers for complex eigenvectors.



The above figure shows two examples initial conditions for such a system: in blue, one selected by the preceding method and lying in the invariant plane spanned by the real and imaginary components of the eigenvectors of the desired modes (1 and 2); in purple, one selected that did not lie in the that plane and required the excitation of the second mode, which in this case was unstable.

10.3.8 Response of linear systems to singularity inputs

Some special cases are worked out in [1, p 353] for singularity inputs $\delta(t)$, $u_s(t)$, and $u_r(t)$.

10.3.9 Systems with repeated eigenvalues

This topic is fully treated in [2, p 250], but not in [1]. Every $n \times n$ matrix has n eigenvalues, and for each distinct eigenvalue λ_i, a linear independent eigenvector $\mathbf{m}_i$ exists. For every eigenvalue λ_i repeated μ_i times (termed *algebraic multiplicity* of λ_i), any number q_i (termed *geometric multiplicity* or *degeneracy* of λ_i) up to and including μ_i of independent eigenvectors may exist: $1 \leq q_i \leq \mu_i$. q_i is equal to the dimension of the null space of $\mathbf{A} - \mathbf{I}\lambda_i$,

$$q_i = n - \text{rank}(A - \lambda_i \mathbf{I}). \tag{123}$$

This gives rise to three cases:

Fully degenerate: $q_i = \mu_i$ In this case, the eigenvalue problem has $q_i = \mu_i$ independent solutions for $\mathbf{m}_i$. So, even though there were not n distinct eigenvalues, n distinct eigenvectors still exist and we can diagonalize or decouple the system as before.

Simple degeneracy: $q_i = 1$ In this case, the eigenvalue problem has $q_i = 1$ independent solutions for $\mathbf{m}_i$. We would still like to construct a basis set of n independent vectors, but they can no longer be eigenvectors, and we will no longer be able to fully diagonalize or decouple the system. There are multiple ways of doing this (e.g. Gram-Schmidt), but the typical and most nearly diagonal way is to construct $\mu_i - q_i$ *generalized eigenvectors* (here also called $\mathbf{m}_i$), which will be included in the modal matrix $\mathbf{M}$ along with the eigenvectors. The generalized eigenvectors are found by solving the usual eigenvalue/vector problem for the first eigenvector $\mathbf{m}_i^1$ corresponding to λ_i, then solving it again with the following equations to find the generalized eigenvectors

$$\begin{aligned}(\mathbf{A} - \lambda_i)\mathbf{m}_i^2 &= \mathbf{m}_i^1 \\ (\mathbf{A} - \lambda_i)\mathbf{m}_i^3 &= \mathbf{m}_i^2 \\ &\vdots\end{aligned}$$

This forms the modal matrix M. The block-diagonal *Jordan form* matrix, analogous to the diagonal $\mathbf{\Lambda}$ is

$$\mathbf{J} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}, \tag{124}$$

which gives the most-decoupled state transition matrix

$$\boxed{\mathbf{\Phi}(t) = \mathbf{M}e^{\mathbf{J}t}\mathbf{M}^{-1}}. \tag{125}$$

General degeneracy: $q_i = 1$ If $1 < q_i < \mu_i$, the preceding method applies, but it may be ambiguous as to which eigenvector the generalized eigenvectors correspond (or how many for each). This can be approached by trial and error or a systematic method presented in [2, p 255].

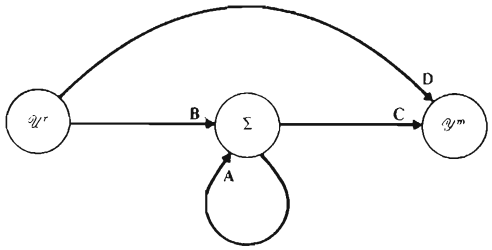
10.4 Controllability & observability

Two very important system properties, *controllability* and *observability* are briefly presented here (for more see [2, p 373]). Two similarity transforms are helpful in transforming the system into forms in which controllability and observability are apparent; they are that using the modal matrix, $\mathbf{x} = \mathbf{M}\mathbf{q}$, and the $\mathbf{QR}$ decomposition. The former leads to a Jordan form of the state equations, the latter to the *Kalman's controllable* and/or *Kalman's observable* form.

The *system representation* or *realization* $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ can be considered as the maps

$$\begin{aligned}\mathbf{A} &: \Sigma \to \Sigma \\ \mathbf{B} &: \mathscr{U}^r \to \Sigma \\ \mathbf{C} &: \Sigma \to \mathscr{Y}^m \\ \mathbf{D} &: \mathscr{U} \to \mathscr{Y}^m\end{aligned}$$

between the different sets $\mathscr{U}^r$, the r-dimensional input set; Σ, the n-dimensional state-space; and $\mathscr{Y}^m$, the m-dimensional output set (see figure below).



10.4.1 Definitions

Controllability [2, p 374] gives the following definition of controllability.

> A linear system is said to be *controllable* at t_0 if it is possible to find some input function (or sequence in the discrete case) $\mathbf{u}(t)$, defined over $t \in \mathscr{T}$, which will transfer the initial state $\mathbf{x}(t_0)$ to the origin at some finite time $t_1 \in \mathscr{T}$, $t_1 > t_0$. That is, there exists some input $\mathbf{u}_{[t_0,t_1]}$, which gives $\mathbf{x}(t_1) = 0$ at a finite time $t_1 \in \mathscr{T}$. If this is true for all initial times t_0 and all initial states $\mathbf{x}(t_0)$, the system is *completely controllable*.

Observability [2, p 375] gives the following definition of observability.

> A linear system is said to be *observable* at t_0 if $\mathbf{x}(t_0)$ can be determined from the output function $\mathbf{y}_{[t_0,t_1]}$ (or output sequence) for $t_0 \in \mathscr{T}$ and $t_0 \leq t_1$, where t_1 is some finite time belonging to $\mathscr{T}$. If this is true for all t_0 and $\mathbf{x}(t_0)$, the system is said to be *completely observable*.

10.4.2 Dependence on model

Both controllability and observability are properties of the specific system representation $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$, which is not unique for a given system. While this is true, a system lacking controllability can typically be granted it with the addition of actuators, just as a system lacking observability can typically be given it by adding sensors.

If there exists a system representation of order n that is both controllable and observable, all system representations of order n are both controllable and observable. If there exists a system representation of order n that is either not controllable or not observable, no system representation exists that is both, some may be controllable but not observable, others may be observable but not controllable, and still others may be neither.

10.4.3 LTI systems with distinct eigenvalues

In this case there is no need to refer to any specific time interval. The system can be decomposed into its Jordan form as

$$\dot{\mathbf{q}} = \mathbf{\Lambda}\mathbf{q} + \mathbf{B}'\mathbf{u} \tag{126}$$

$$\mathbf{y} = \mathbf{C}'\mathbf{q} + \mathbf{D}\mathbf{u} \tag{127}$$

where $\mathbf{q} = \mathbf{M}^{-1}\mathbf{x}$, $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}$ is the modal matrix of eigenvectors, $\mathbf{B}' = \mathbf{M}^{-1}\mathbf{B}$, and $\mathbf{C}' = \mathbf{C}\mathbf{M}$. This fully decoupled form allows us to easily ascertain the controllability and observability of the system representation.

Controllability criterion 1

> The constant coefficient system, for which $\mathbf{A}$ has distinct eigenvalues, is completely controllable if and only if there are no zero rows of $\mathbf{B}' = \mathbf{M}^{-1}\mathbf{B}$.

Observability criterion 1

> The constant coefficient system, for which $\mathbf{A}$ has distinct eigenvalues, is completely observable if and only if there are no zero columns of $\mathbf{C}' = \mathbf{C}\mathbf{M}$.

10.4.4 LTI systems with arbitrary eigenvalue

This case can be handled using a variation of the distinct eigenvalues criteria, but the method is lengthy, since it involves finding the modal decomposition into Jordan form. The gist of it is that for a decoupled system mode to be controllable, it must either have a direct connection to the input (nonzero row of $\mathbf{B}'$) or be coupled to another mode that has one. So nonzero rows of $\mathbf{B}'$ can be tolerated if they are not the last row associated with a given Jordan block. Similarly, a system is observable if the first column associated with a given Jordan block is not identically zero.

The more useful criteria follow.

Controllability criterion 2

> A constant coefficient linear system is completely controllable if and only if the $n \times rn$ matrix of
>
> $$\mathbf{P} = \left[\mathbf{B}|\mathbf{A}\mathbf{B}|\mathbf{A}^2\mathbf{B}|\cdots|\mathbf{A}^{n-1}\mathbf{B}\right], \tag{128}$$
>
> called the *controllability matrix*, has rank n. The number of partitions of P required to achieve rank n is termed the *controllability index*, with lower indices implying "better" controllability.

Observability criterion 2

A constant coefficient linear system is completely observable if and only if the $n \times mn$ matrix

$$\mathbf{Q} = \left[\overline{\mathbf{C}}^{\mathsf{T}} | \overline{\mathbf{A}}^{\mathsf{T}}\overline{\mathbf{C}}^{\mathsf{T}} | \overline{\mathbf{A}^2}^{\mathsf{T}}\overline{\mathbf{C}}^{\mathsf{T}} | \cdots | \overline{\mathbf{A}}^{n-1\,\mathsf{T}}\overline{\mathbf{C}}^{\mathsf{T}}\right], \quad (129)$$

called the *observability matrix*, has rank n. The number of partitions of Q required to achieve rank n is termed the *observability index*, with lower indices implying "better" observability. Note that the complex conjugates are typically not required.

10.4.5 Stabilizability & detectability

Weaker versions of controllability and observability are *stabilizability* and *detectability*. The are defined as follows.

Stabilizability

A linear system is said to be stabilizable if all its unstable modes, if any, are controllable.

Detectability

A linear system is said to be detectable if all of its unstable modes, if any, are observable.

11 The transfer function[1, p 395]

The *transfer function* is a representation of input-output dynamics of linear systems. There are different methods for finding the system transfer function, but the one presented here is called the *generalized exponential method* (note that other methods give slightly different interpretations of the transfer function). It is developed in terms of the *particular solution* of the system to an *exponential input*

$$u(t) = U(s)e^{st} \quad (130)$$

where the complex variable is $s = \sigma + j\omega$ and the amplitude $U(s)$ is complex in general. It can also be written

$$u(t) = U(s)e^{(\sigma+j\omega)t} = U(s)e^{\sigma t}(\cos\,\omega t + j\,\sin\,\omega t), \quad (131)$$

which shows that this form of input covers a broad range of inputs of interest, including decaying sinusoidal waveforms. This derivation provides a basis for determining the steady-state response characteristics of periodic waveforms.

11.1 SISO systems

Given the classical representation (note this is different from Equation (34)

$$a_n\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y$$
$$= b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots + b_1\frac{du}{dt} + b_0 u, \quad (132)$$

and the assumed particular solution

$$y_p(t) = Y(s)e^{st} \qquad (Y(s) \in \mathbb{C}), \quad (133)$$

the transfer function is defined as the ratio of the response amplitude $Y(s)$ to the input amplitude $U(s)$:

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \quad (134)$$

The output can now be written as

$$y_p(t) = H(s)U(s)e^{st}. \quad (135)$$

11.2 Relation to the transfer operator $H\{\}$

The only differences for SISO systems between $H(s)$ and $H\{\}$ are in interpretation. $H(s)$ is an algebraic quantity describing the system particular response to an exponential input and can be manipulated using linear algebra, whereas $H\{\}$ is an operator that is independent of the form of the input and implies a causal relationship between system input and output.

The complex variable s and the differential operator $S\{\}$ can be used interchangeably for LTI systems.

11.3 Poles and zeros

The numerator and denominator of Equation (134) can be factored into the form

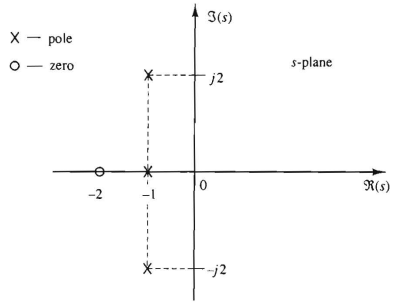
$$H(s) = \frac{N(s)}{D(s)} = K\frac{(s-z_1)(s-z_2)\ldots(s-z_{m-1})(s-z_m)}{(s-p_1)(s-p_2)\ldots(s-p_{n-1})(s-p_n)} \quad (136)$$

where $K = b_m/a_n$ is the gain constant.

Confusion can arise from definitions of poles and zeros being inconsistent between texts. [1] uses the definition that sets the numerator and denominator of (136) equal to zero. Another takes the poles and zeros to be the values of s such that the limit of (136) as s approaches a pole or zero to go to ∞ or 0. These two approaches lead to different poles and zeros in the case of *pole-zero cancellation*, see Sec 11.3.5.

11.3.1 The pole-zero plot

Some system properties can be discovered from the plot of the poles and zeros on the complex s-plane as shown in the following figure.



11.3.2 System poles and the homogeneous response

Although developed here from the particular response, the transfer function still describes the entire differential equation of the system, so the homogeneous response is embedded in the transfer function. The characteristic equation is the denominator of the transfer function set equal to zero, so the system eigenvalues are equal to the system poles (including canceled ones, see Sec 11.3.5). This gives the homogeneous solution

$$y_h(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t} = \sum_{i=1}^{n} C_i e^{p_i t} \quad (137)$$

where C_i is found from the initial conditions. The locations of the poles define the n components of the homogeneous response as follows:

1. $\boldsymbol{p_i = -\sigma}$ **:** the component $Ce^{-\sigma t}$ is a decaying exponential,
2. $\boldsymbol{p_i = 0}$ **:** the component C is a constant amplitude,
3. $\boldsymbol{p_i = +\sigma}$ **:** the component $Ce^{\sigma t}$ is an increasing exponential,
4. $\boldsymbol{p_i = -\sigma \pm j\omega}$ **:** the component $Ae^{-\sigma t}\sin(\omega t + \phi)$ is a decaying sinusoid,
5. $\boldsymbol{p_i = \pm j\omega}$ **:** the component $A\sin(\omega t + \phi)$ is a sinusoid, and
6. $\boldsymbol{p_i = +\sigma \pm j\omega}$ **:** the component $Ae^{\sigma t}\sin(\omega t + \phi)$ is an increasing sinusoid.

Note that the larger the distance along the real-axis from the origin, the greater the exponential decay or increase σ. Also, the larger the distance along the imaginary-axis from the origin, the greater the frequency of oscillation ω.

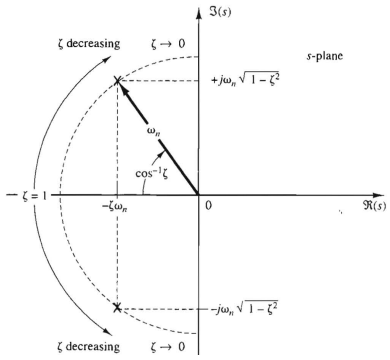
For second-order systems,

$$p_1,\, p_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \quad (138)$$

For underdamped second-order systems,

$$p_1,\, p_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2}. \quad (139)$$

The following figure shows the pole locations and their relationships to ω_n and ζ.



11.3.3 System stability in terms of poles

Asymptotic stability An *asymptotically stable* is a system that, with the input identically zero, for all initial conditions, in the limit as time approaches infinity, the state asymptotically approaches zero. Since pole-zero cancellation does not play a role in initial-condition response, all the systems eigenvalues are identical to its poles, and *vice-versa*. For asymptotic stability, all its poles must have negative real parts. A marginally stable system requires that at least one pole have zero real part, and no poles with positive real parts. An unstable system is one that has at least one pole with a positive real part.

Bounded-input-bounded-output stability (BIBO) A system is *BIBO* if, with initial conditions all zero, no bounded input causes an unbounded output. Pole-zero cancellation can play a role here, since a transfer function governs the input-output relationship, and an unstable pole could be canceled by a corresponding zero. While the transfer function may have no unstable poles, due to pole-zero cancellation there remains the possibility of an unstable mode being excited by the input. If the system is not asymptotically stable, it can still be BIBO stable iff (1) its B matrix is such that the no unstable mode is excited by any input $u(t)$ or (2) its C matrix is such that the unstable mode is not "noticed" (although this is not practical).

11.3.4 Interpretation of poles and zeros

Poles and zeros affect the system output in different ways, depending on the s-plane location of the pole or zero, the type of input present (sinusoidal, etc.), and the type of output that is of interest (transient, steady-state, etc.).

Poles Poles, when there is no pole-zero cancellation (see Sec 11.3.5), are identical to the system eigenvalues, and are always a subset of them. This means that, along with the initial conditions, they always completely determine the homogeneous response of the system, as in (137). Note that this remains true *even when pole-zero cancellation occurs* (see Sec 11.3.5).

Poles completely determine system stability. Depending on the given definition of poles and zeros, canceled poles and zeros may not be considered poles and zeros, but regardless, even these canceled poles govern stability due to initial conditions (so be sure to include them in an analysis of stability).

A periodic input can be described (by a Fourier series) as a sum of sinusoidal inputs and, by superposition, we can sum the responses of the system to each sinusoidal input. This means that the frequency response function $H(j\omega)$ governs the steady-state response. The poles affect such a system's response to periodic inputs by the manner in which they govern the bode plot construction.

Zeros Zeros are relevant only to response of the system to the *input*, since initial condition responses are unaffected by the zeros. For periodic inputs in steady-state, zeros affect the frequency response function in a manner analogous to the poles.

11.3.5 Pole-zero cancellation

Pole-zero cancellation occurs occasionally. Recall that this occurs for a transfer function, which describes the input-output relationship, and tells us nothing of the system's response to initial conditions. The canceled pole remains a component of the homogeneous response to initial conditions, but is considered not to affect the input-output relationship (think about it in terms of Laplace transforms of the input-output differential equation - there is no pole-zero cancellation for the initial condition term).

11.4 Transfer functions of interconnected systems

For systems in series and parallel, the transfer functions can be combined. Two systems in series (cascading series), meaning the output of the first is the input of the second, *if it can be assumed that the second does not "load" the first* (i.e. the first system acts as an ideal source to the second), can combine transfer functions in the following way

$$H(s) = H_1(s)H_2(s). \tag{140}$$

For two systems in parallel, meaning that they share the same input and their outputs are summed, the combined transfer function can be written as

$$H(s) = H_1(s) + H_2(s). \tag{141}$$

11.5 State-space to transfer functions

The *transfer function matrix* can be constructed from the state equations to be

$$\mathbf{H}(s) = \frac{\mathbf{C}\ \text{adj}\ (s\mathbf{I}-\mathbf{A})\mathbf{B} + \mathbf{D}\ \det\ (s\mathbf{I}-\mathbf{A})}{\det\ (s\mathbf{I}-\mathbf{A})}. \tag{142}$$

The structure of $\mathbf{H}(s)$, the $m \times r$ matrix, is

$$\mathbf{H}(s) = \begin{bmatrix} H_{11}(s) & \dots & H_{1r}(s) \\ \vdots & \ddots & \vdots \\ H_{m1}(s) & \dots & Hmr(s) \end{bmatrix} \tag{143}$$

where $H_{ij}(s)$ is the transfer function from the jth input to the ith output.

For SISO systems,

$$H(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}, \tag{144}$$

which can be calculated using Equation (142).

11.6 Minimal realizations [2, p 408]

Definition

> Of all possible realizations of $\mathbf{H}(s)$, $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is said to be an *irreducible* (or minimum) *realization* if the associated state space has the smallest possible dimension $\dim(\Sigma)$.

A minimal realization is both completely controllable and completely observable. In the case of a scalar transfer function, the minimum dimension required is equal to the order of the denominator of the transfer function *after all common pole-zero cancellations are made.*

A realization obtained from the methods presented earlier, linear graphs, are the most complete descriptions of the internal system. They may not be fully controllable or observable, but this is not always important. A minimal realization makes most sense when the transfer function (scalar or otherwise) is measured, which describes the system with input and output and does not describe the internal system. A minimal realization of this transfer function would not lose any information in this case.

It is instructive to start with a state space realization of dimension n, find the corresponding transfer function $\mathbf{H}(s)$, and find a minimal realization of this of dimension n_1. If $n = n_1$, the system is completely characterized by $\mathbf{H}(s)$, it is both completely controllable and observable, and its poles and eigenvalues are the same. If $n > n_1$, then the information about $n - n_1$ modes was lost (and it could have been unstable, who knows). For a scalar transfer function, the order of the irreducible realization is the degree of the denominator after all common pole/zero pairs are canceled.

11.6.1 transmission zeros

Transmission zeros are values of z for which

$$\text{rank}\begin{bmatrix} \mathbf{A}-z\mathbf{I} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} < \text{rank}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}. \tag{145}$$

12 Impedance-based modeling methods [1, p 422]

This chapter is primarily covered in Professor Joseph Garbini's Appendix A, but I would like to make a few comments here.

12.1 Input impedance (impedance of system)

When an ideal source is connected to a system, either its across- or through-variable is given, and the other is a *function of the system* (e.g. back emf, current "draw") and the given source variable. For an across-variable source, $F_{in}(s) = Y(s)V_{in}(s)$ where $Y(s)$ is the *input admittance.* For a through-variable source, $V_{in}(s) = Z(s)F_{in}(s)$, where $Z(s)$ is the *input impedance.* Ignoring causality, $Y(s) = 1/Z(s)$. This is usually just called the impedance of the system, since this is the impedance that the source "sees" (drives).

12.2 Quick transfer-function generation

This method can be used to either to obtain a general expression for transfer functions between system inputs to outputs, or as a quick and dirty method to obtain a transfer function between a specific input and output. For the more general method, see [1, p 434]. When reducing the system, *be sure to retain those features required for the transfer function* (e.g. don't collapse a node if its across-variable is of interest to the transfer function). Garbini, in the appendix, gives two main "tricks" to reduce a system using impedances and obtain a specific transfer function: across- and through-variable dividers, and Thevenin and Norton equivalent models.

12.2.1 Across- and through-variable dividers

These are presented in Appendix A. The strategy is to reduce a system to one or the other model, so the transfer function can be found using those models. Sometimes the transfer function desired is of the form $T(s) = \frac{V_i}{F_s}$: the elemental variable desired is not matched with the source (across vs through or vice versa).

This can be solved using the relevant elemental relationship (e.g. $\frac{V_i}{F_s} = \frac{\frac{1}{C_i s} F_i}{F_s}$).

12.2.2 Thevenin & Norton equivalent models

If the across- or through-variable dividers are unable to allow the desired transfer function to be found, the source equivalent methods will be of great help.

The idea for Thevenin models is that any linear system excited by a single source (*across or through*) and driving an external load Z_L may be modeled as a single across variable source V_s (Garbini's V_e) connected in series with a single impedance element $Z_{out}(s)$ (Garbini's Z_e).

The idea for Norton models is that any linear system excited by a single source (*across or through*) and driving an external load Z_L may be modeled as a single through variable source F_s (Garbini's F_e) connected in parallel with a single impedance element $Z_{out}(s)$ (Garbini's Z_e).

For construction of these models, see [1, p 441] and *follow the procedure*. There arise some situations where the transfer function desired can make use of transforming between Thevenin and Norton models.

12.3 Transducing elements

These can be dealt with using the following method. For transformers, the effective impedance at port 1 of the impedance Z_3 connected across port 2 is

$$Z_1(s) = (\text{TF})^2 Z_3(s). \tag{146}$$

For gyrators, the same quantity is

$$Z_1(s) = (\text{GY})^2 \frac{1}{Z_3(s)}. \tag{147}$$

13 Sinusoidal frequency response of linear systems [1, p 453]

In this section we will investigate the response of a system due to inputs of the form

$$u(t) = A \sin(\omega t + \psi) \tag{148}$$

where A is amplitude, ω is the angular frequency ($\omega = 2\pi/T$, where T is the *period*), and ψ is the phase.

Here we deal only with the *steady-state response*, since the transient terms of the sinusoidal response (due to ICs) die out quickly.

We begin with the steady-state (particular) solution to an exponential input $u(t) = U(s)e^{st}$,

$$y_p(t) = Y(s)e^{st} = H(s)U(s)e^{st}, \tag{149}$$

where $H(s)$ is the transfer function and $U(s)$ is the *complex input amplitude*. Since we only desire sinusoidal response, $s \Rightarrow j\omega$. This leads to the expression for the *complex output amplitude* $Y(j\omega)$ as a function of a gain and the complex input amplitude:

$$Y(j\omega) = H(j\omega)U(j\omega) \tag{150}$$

where $H(j\omega)$ is defined as the *frequency response* of the system. From the transfer function

$$H(j\omega) = H(s)|_{s=j\omega}. \tag{151}$$

The *magnitude* or *gain* of the frequency response is

$$|H(j\omega)| = \sqrt{[\text{Re}(H(j\omega))]^2 + [\text{Im}(H(j\omega))]^2)} \tag{152}$$

and the *phase angle* of the frequency response is

$$\phi(j\omega) = \tan^{-1}\left(\frac{\text{Im}(H(j\omega))}{\text{Re}(H(j\omega))}\right). \tag{153}$$

The steady-state solution $y_{ss}(t)$ for a system with transfer function $H(s)$ (frequency response $H(j\omega)$) to a sinusoidal input $u(t)$ is

$$y_{ss}(t) = A|H(j\omega)| \sin(\omega t + \psi + \phi(j\omega)). \tag{154}$$

This means that for a sinusoidal input, the system responds (in steady-state) with a sinusoidal output at the same frequency with amplitude scaled by $|H(j\omega)|$ and phase-shifted $\phi(j\omega)$. $|H(j\omega)|$ is the ratio of the output amplitude to the input amplitude, sometimes called *gain* of the system, as a function of input frequency ω. Systems that respond to low frequencies but attenuate high frequencies are called *low-pass filters*, while systems that response to high frequencies but attenuate low frequencies are called *high-pass filters*.

13.1 1st- & 2nd-order systems

13.1.1 First-order systems

First-order systems of the form

$$\tau \frac{dy}{dt} + y = K_0 u(t), \tag{155}$$

where K_0 is a constant, has transfer function

$$H(s) = \frac{K_0}{\tau s + 1}. \tag{156}$$

The frequency response function is

$$H(j\omega) = \frac{K_0}{j\omega\tau + 1}, \tag{157}$$

and its magnitude and phase are

$$|H(j\omega)| = \frac{K_0}{\sqrt{(\omega\tau)^2 + 1}} \tag{158}$$

$$\phi(j\omega) = \tan^{-1}(-\omega\tau). \tag{159}$$

From Equation (154), the steady-state response to a sinusoidal input is

$$y_{ss}(t) = A\frac{K_0}{\sqrt{(\omega\tau)^2 + 1}} \sin(\omega t + \psi + \tan^{-1}(-\omega\tau)). \tag{160}$$

13.1.2 Second-order systems of the usual form

Second-order systems of the form

$$\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = K_0 u(t) \tag{161}$$

where K_0 is a constant, has transfer function

$$H(s) = \frac{K_0}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{162}$$

The frequency response function is

$$H(j\omega) = \frac{K_0/\omega_n^2}{(1 - \frac{\omega^2}{\omega_n^2}) + j(2\zeta\frac{\omega}{\omega_n})}, \tag{163}$$

and its magnitude and phase are

$$|H(j\omega)| = \frac{K_0/\omega_n^2}{\sqrt{(1 - \frac{\omega^2}{\omega_n^2})^2 + (2\zeta\frac{\omega}{\omega_n})^2}} \tag{164}$$

$$\phi(j\omega) = \tan^{-1}\frac{-2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2}. \tag{165}$$

The maximum (peak) magnitude occurs at the *resonance frequency*

$$\omega_P = \omega_n\sqrt{1 - 2\zeta^2} \qquad \forall\ \zeta \leq \sqrt{2}/2 = .707 \tag{166}$$

since for $\zeta \geq .707$ no peak occurs and the magnitude is monotonically decreasing for increasing ω. Note that this is only valid for systems of the form (161). It was derived from taking the derivative of the magnitude of the frequency response function wrt frequency, setting equal to zero, and solving for ω. The solution varies if the transfer function is of another form, for instance, if the transfer function has a zero. The magnitude at this frequency is

$$|H(j\omega)| = \frac{K_0/\omega_n^2}{2\zeta\sqrt{1 - \zeta^2}}. \tag{167}$$

13.2 Bode plots

Bode plots are magnitude and phase angle plots as functions of input (driving) frequency ω. They are typically graphed on decibel (dB) and degree (°) vertical axes and logarithmic horizontal axes. Decibels are found by the formula

$$Q = 10 \log_{10}\left(\frac{\mathcal{P}}{\mathcal{P}_{ref}}\right) \text{ dB} = 20 \log_{10}\left(\frac{A}{A_{ref}}\right) \text{ dB}. \quad (168)$$

$|H(j\omega)|$ is the ratio of the amplitude of the sinusoidal input to the amplitude of the sinusoidal output, so in dB,

$$|H(j\omega)|_{\text{dB}} = 20 \log_{10}|H(j\omega)| \text{ dB}. \quad (169)$$

The following table is helpful for quick conversions.

dB - power - amplitude relationships

dB	$\mathcal{P}_{out}/\mathcal{P}_{in}$	A_{out}/A_{in}
-40	0.0001	0.01
-30	0.001	0.03162
-20	0.01	0.1
-10	0.1	0.3162
-6	0.25	0.5
-3	0.5	0.7071
0	1	1
3	2	1.414
6	4	2
10	10	3.162
20	100	10
40	10000	100

Another common measure is the *decade*, which is synonymous with the "*order of magnitude*" - a factor of 10.

13.2.1 Asymptotic Bode plots from transfer function

This method for hand-drawing Bode plots is best-suited when the transfer function is known (see Section 13.2.2 when the pole-zero plot is known).

The general method is to build a higher-order bode plot from easier-constructed lower-order plots. The magnitude Bode-plot characteristics of some of these are in the following table.

Asymptotic properties for magnitude Bode plot

Description ($\times$ = pole, $\circ$ = zero)	Transfer function	Break frequency (rad/s)	High-ω slope (dB/decade)
Const gain	K	–	0
$\times$ at origin	$1/s$	–	-20
$\circ$ at origin	s	–	20
Real $\times$	$1/(\tau s + 1)$	$1/\tau$	-20
Real $\circ$	$\tau s + 1$	$1/\tau$	20
Conj $\times$'s	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	ω_n	-40
Conj $\circ$'s	$\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$	ω_n	40

The following procedure can be followed to construct the Bode magnitude plot:

1. factor the numerator and denominator of $H(s)$ into constant, first-order, and second-order terms s.t. each term has a corresponding entry in the above table,
2. identify the break frequency associated with each factor (from table),
3. plot the asymptotic of *each* of the factors on dB-log or log-log axes,
   (a) draw horizontal line (unless there is a pole or zero at the origin) at the constant-gain value up to the break frequency, and
   (b) draw (linear on log) line using the "high-ω" slope (negative if pole, positive if zero)
4. graphically add the component plots (multiplication is addition in log-scale: $\log(ab) = \log a + \log b$), and
5. "round out" the corners using known values at break frequencies (± 3 dB for first-order sections, and dependent on ζ for quadratic factors).

Phase plots can be constructed by using a similar method. Here is a table for phase-plot properties.

Asymptotic properties for phase Bode plot

Description ($\times$ = pole, $\circ$ = zero)	Break frequency ω_B (rad/s)	Phase @ ω_B (°)	High-ω phase (°)
Const gain	–	–	0
$\times$ at origin	–	–	-90
$\circ$ at origin	–	–	90
Real $\times$	$1/\tau$	-45	-90
Real $\circ$	$1/\tau$	45	90
Conj $\times$'s	ω_n	-90	-180
Conj $\circ$'s	ω_n	90	180

The procedure for plotting the phase is as follows (first two steps redundant if magnitude plot is already finished):

1. factor the numerator and denominator of $H(s)$ into constant, first-order, and second-order terms s.t. each term has a corresponding entry in the above (magnitude) table,
2. identify the break frequency associated with each factor (from table),
3. plot *each* component phase plot
   (a) if the component is one of the *first three* in the table, draw the constant line and skip the rest of the sub-steps
   (b) plot the break frequency ω_B point (see above table),
   (c) if the component is a *real pole or zero*,
      i. plot the horizontal zero-phase up to *one decade below* ω_B,
      ii. plot the high-frequency asymptote (see above table) *after one decade above* ω_b,
      iii. linearly connect these two lines, drawing a line through the "Phase @ ω_B",
   (d) if the component is a conjugate pair, inspect phase plot shapes in [1, p 476, Figure 14.15] based on ζ, but note that the low-ω, "Phase @ ω_B," and "High-ω phase" will all be satisfied (see above table),
4. graphically add each component (here the phases are, in fact, additive, and on a linear scale).

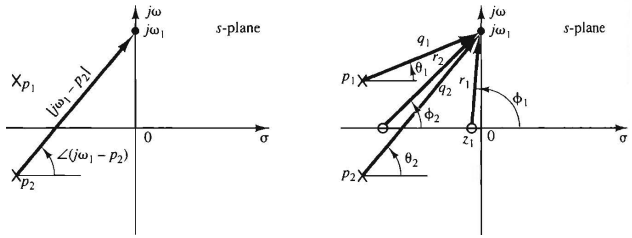
13.2.2 Bode plots from pole-zero plots

Given a factored transfer function, the poles and zeros, or just the pole-zero plot, much can be said about the magnitude and phase response, and a magnitude Bode plot can be quickly sketched. If each factor in a factored frequency response function, each corresponding to a pole or zero, is considered as a vector from the pole or zero location to the location of $j\omega$, which is the input frequency - located on the vertical axis, then the magnitude and phase of the frequency response function are

$$|H(j\omega)| = K\frac{r_1 \cdot r_2 \cdots r_m}{q_1 \cdot q_2 \cdots q_n} \quad (170)$$

$$\angle H(j\omega) = (\phi_1 + \phi_2 + \cdots + \phi_m) - (\theta_1 + \theta_2 + \cdots + \theta_n) \quad (171)$$

where $r_i = |(j\omega - z_i)|$, $q_i = |(j\omega - p_i)|$, K is a constant that cannot be determined from the pole-zero plot, θ_i and ϕ_i are the angles pole and zero vectors (respectively) make to the horizontal.



The following observations can be made from these plots:

- for a real pole:
  - for low-ω, the magnitude approaches a finite value,
  - as ω increases, the magnitude and phase decrease,
  - for high-ω, the magnitude $\Rightarrow 0$, and phase $-\pi/2$;
- for a real zero:
  - for low-ω, the magnitude approaches a finite value,
  - as ω increases, the magnitude and phase increase,
  - for high-ω, the magnitude $\Rightarrow \infty$, and phase $\pi/2$;
- for complex conjugate poles:
  - for low-ω, the magnitude and phase approaches a finite value,
  - as ω passes nearest the poles, the magnitude reaches a maximum (resonance),
  - as ω increases further, the magnitude and phase decrease,
  - for high-ω, the magnitude $\Rightarrow 0$, and phase $-2\pi/2 = -\pi$;
- for complex conjugate poles:
  - for low-ω, the magnitude and phase approaches a finite value,
  - as ω passes nearest the zeros, the magnitude reaches a minimum,
  - as ω increases further, the magnitude and phase increase, and
  - for high-ω, the magnitude $\Rightarrow \infty$, and phase $2\pi/2 = \pi$.

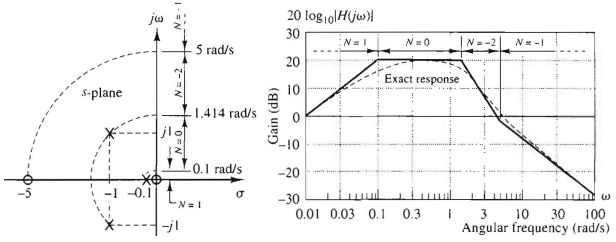
The following generalizations can be made.

- If a system has more poles than zeros, as the frequency becomes large, the magnitude tends to zero.
- If a system has more zeros than poles, as the frequency becomes large, the magnitude tends to infinity (which cannot happen in a physical system).
- If a system has a pair of complex conjugate poles near the imaginary axis, a peak magnitude is seen as the frequency approaches them. If the poles are on the imaginary axis, the magnitude is infinite at that frequency.
- If a system has a pair of complex conjugate zeros near the imaginary axis, a minimum (notch) magnitude is seen as the frequency approaches them. If the poles are on the imaginary axis, the magnitude is zero at that frequency.
- A pole at the origin (pure integrator) implies an infinite magnitude at DC-frequency.
- A zero at the origin (pure differentiator) implies a zero gain DC-frequency.

Constructing magnitude Bode plots
The pole-zero plot provides a good method for constructing a magnitude Bode plot. The constant K from Equation (170) cannot be determined from the pole-zero plot, so the Bode plot must be scaled (shifted up or down for log-plot) appropriately. The key idea is that the vector drawn from the origin to each pole or zero corresponds to the break frequency (the magnitude of the vector) and the phase angle (the angle of the vector).

Break the pole-zero plot into radial regions with boundaries between the regions being the radial positions of the poles and zeros. These radial positions correspond to (equal!) break frequencies in the Bode magnitude plot (at this point, dash-line in the break frequencies). Starting from the origin, move outward "picking up" poles and zeros as you go. The slope of the Bode plot in each region depends on how many poles and zeros you have carried to that region. If you pass a pole, the slope decreases by 20 dB; if you pass a zero, the slope increases by 20 dB. The formula for the slope in each region as a function of the number of poles P and zeros Z between it and the origin is

$$\text{slope} = 20(Z - P) \text{ dB/decade}. \tag{172}$$



14 Frequency domain methods [1, p 500]

In the section we develop methods of determining the system response to more general input functions. We will describe inputs as sums (or integrals) of many sinusoidal inputs and develop, based on the principle of superposition (sum of inputs gives sum of individual outputs), methods for finding system responses.

14.1 Fourier analysis of periodic waveforms (Fourier Series)

A periodic function is one that satisfies $f(t) = f(t + nT)\ \forall n = \pm 1, \pm 2, \ldots$ where T is the *period* of the function. The *fundamental frequency* ω_0 is defined as $\omega_0 \equiv 2\pi/T$. Waveforms are called *harmonics* of each other if their fundamental frequencies (or periods) are an **integer** ratio (meaning the greater divided by the smaller equals an integer).

The *Fourier series* is a description of a *periodic* function as a weighted sum of harmonic sinusoidal components. The nth harmonic component of the series can be written

$$f_n(t) = a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \tag{173}$$
$$= \mathcal{A}_n \sin(n\omega_0 t + \phi_n) \tag{174}$$

where the conversions between each representation are

$$a_n = \mathcal{A}_n \sin\phi_n \quad \text{and} \quad b_n = \mathcal{A}_n \cos\phi_n \tag{175}$$

$$\mathcal{A}_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \phi_n = \tan^{-1}\frac{a_n}{b_n}. \tag{176}$$

The Fourier series representation of a function $f(t)$ subject to certain conditions [1, p 509] is

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \tag{177}$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\mathcal{A}_n \sin(n\omega_0 t + \phi_n). \tag{178}$$

A third, equivalent, complex representation is

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{179}$$

where

$$F_n = \frac{1}{2}(a_n - jb_n) \tag{180}$$

$$F_{-n} = \frac{1}{2}(a_n + jb_n). \tag{181}$$

So we have a representation of any periodic function as a weighted sum of sinusoids. To determine the weights (coefficients), the following formulas can be used. For the real representations,

$$a_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\cos(n\omega_0 t)dt, \tag{182}$$

$$b_n = \frac{2}{T}\int_{t_1}^{t_1+T} f(t)\sin(n\omega_0 t)dt, \tag{183}$$

and a_0 can be computed separately from Equation (182). For the complex representation,

$$F_n = \frac{1}{T}\int_{t_1}^{t_1+T} f(t)e^{-jn\omega_0 t}dt, \tag{184}$$

F_{-n} is the complex conjugate of F_n, and F_0 can be computed *separately* (with $n = 0$) from (184).

14.1.1 Properties of the Fourier series

See [1, p 509] for more, but a few properties are listed here.

- **Linearity** If the Fourier series components of two periodic functions $g(t)$ and $h(t)$ with identical periods T are G_n and H_n, then, if a new function $f(t)$ is defined as a linear combination of $g(t)$ and $h(t)$,

$$f(t) = ag(t) + bh(t), \tag{185}$$

the Fourier components of $f(t)$ are

$$F_n = aG_n + bH_n. \tag{186}$$

- **Even and Odd Functions** If the function is even, then all $b_n = 0$. If a function is odd, then all $a_n = 0$. Note that if a function would be odd if it was shifted up or down by some DC value, assume $a_n = 0$, but shift the result by the same shift (see next property).
- **Interpretation of the zero-frequency (DC) term a_0 or F_0** This term is simply the average value of the function over a period. If a function is shifted up or down, this is the only term in the series that is effected, and it is shifted by the same average or DC value.

14.1.2 Line spectra

For a waveform decomposed into Fourier components, it is natural to describe the waveform as a *line spectrum*. This is most commonly represented as a magnitude and phase plot versus frequency, using (176) or simply the absolute values and phases of the complex coefficients $|F_n|$ and $\angle F_n$. For the complex case, sometimes the "magnitude" plot is of the coefficient F_n' in front of the exponential of $F_n = F_n' e^{\phi_n}$ where $|F_n| = |F_n'|$.

The complex representation gives rise to a *two-sided spectrum*, meaning that negative n gives rise to negative frequency components. The real representation gives a *one-sided spectrum*, for which only positive n and frequencies arise. For conversion between one- and two-sided spectra, use (180) and (181). To convert a two-sided magnitude line spectrum to a one one-sided, the plot is merely "folded" over and doubles its positive half (since it is symmetric about the magnitude axis) or $\mathcal{A}_n = 2|F_n|$.

14.2 Response of linear systems to periodic inputs

Given a system with frequency response $H(j\omega)$ and an input $u(t)$ of the form of Equations (177) or (178), the output of each component of the input is

$$y_n(t) = |H(jn\omega_0)|\mathcal{A}_n \sin[n\omega_0 t + \phi_n + \angle H(Jn\omega_0)]. \tag{187}$$

By the principle of superposition,

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{188}$$

Given the same situation but with the input in the form of Equation (179),

$$y_n(t) = H(jn\omega_0)F_n e^{jn\omega_0 t} \tag{189}$$

and, once again using superposition,

$$y(t) = \sum_{n=-\infty}^{\infty} y_n(t). \tag{190}$$

14.3 Fourier analysis of transient waveforms (Fourier Transforms)

Some *aperiodic* (transient) functions, which do not have Fourier series representations, can be described by a *Fourier Transform* (also called the *spectrum*). These functions must be limited in time, occur only once, and decay to zero as time becomes large. A Fourier series of a periodic extension of the function is used in the development of the periodic extension, then the interval between occurrences of the function taken to infinity, which gives the Fourier transform.

The Fourier transform is defined by the following *Fourier transform pair*:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \tag{191}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega \tag{192}$$

and can also be written in terms of the Fourier transform operator $\mathcal{F}$,

$$F(j\omega) = \mathcal{F}\{f(t)\} \tag{193}$$

$$f(t) = \mathcal{F}^{-1}\{F(j\omega)\}. \tag{194}$$

See [1, pp 528-31] for the properties of the Fourier transform, including *existence*, *linearity*, *even and odd functions*, *time-shifting*, *waveform energy*, and *Fourier transforms of the derivative of a function*.

14.4 Fourier transform-based properties of linear systems

14.4.1 Response of linear systems to aperiodic inputs

The output spectrum is the input spectrum scaled by the frequency response function:

$$Y(j\omega) = H(j\omega)U(j\omega). \tag{195}$$

This leads to the following procedure for finding the output of a system due to an aperiodic input:

1. compute the Fourier transform of the input

$$U(j\omega) = \mathcal{F}\{u(t)\},$$

2. form the output spectrum as the product

$$Y(j\omega) = H(j\omega)U(j\omega), \text{ and}$$

3. compute the inverse Fourier transform

$$y(t) = \mathcal{F}^{-1}\{Y(j\omega)\}.$$

14.4.2 Relationship between $H(j\omega)$ and $h(t)$

The Fourier transform (spectrum) of the impulse response $h(t)$ of a system is the frequency response function $H(j\omega)$:

$$H(j\omega) = \mathcal{F}\{h(t)\} \tag{196}$$

$$h(t) = \mathcal{F}^{-1}\{H(j\omega)\}. \tag{197}$$

This completely characterizes the system in the frequency or time domain, respectively.

This has important implications for measuring a system frequency response function $H(j\omega)$. We can provide an impulse input (strike it with a hammer at the input) and measure $y(t)$ at the output, then take the Fourier transform to obtain $H(j\omega)$ (or transfer function), with which we can predict the output of a function to a variety of inputs.

14.4.3 Convolution

Convolution in the time domain (see Section 8.4), is simply multiplication in the frequency domain:

$$\mathcal{F}\{f(t) \star g(t)\} = F(j\omega)G(j\omega). \tag{198}$$

Conversely, convolution in the frequency domain, is a scaled multiplication in the time domain:

$$\mathcal{F}^{-1}\{F(j\omega) \star G(j\omega)\} = 2\pi f(t)g(t). \tag{199}$$

14.4.4 Frequency response of interconnected systems

If two linear systems are connected in cascade (series), and *provided the connection does not effect the output of the first system*, the overall frequency response is

$$H(j\omega) = H_1(j\omega)H_2(j\omega). \tag{200}$$

If two linear systems are connected in parallel (share the input and their outputs sum), the overall frequency response is

$$H(j\omega) = H_1(j\omega) + H_2(j\omega). \tag{201}$$

14.5 Laplace Transforms

Many functions do not have Fourier transforms, including the unit step and ramp functions. The *Laplace transform* is a generalized form of the Fourier transform that exists for a much broader range of functions.

The Laplace transform multiplies by a weighting function $w(t) = e^{-\sigma t}$ to drive the integral to zero. Now the one-side transform (assuming $f(t) = 0\ \forall t < 0$) pair is

$$F(s) = \int_0^{\infty} f(t)e^{-st}dt \tag{202}$$

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \tag{203}$$

where $s = \sigma + j\omega$. The *region of convergence* (ROC) of the integral in the s-plane is an important quantity for each Laplace transform. If the imaginary axis is in the ROC, the Fourier transform exists for that function. In operator form

$$\mathcal{L}\{f(t)\} = F(s) \tag{204}$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t). \tag{205}$$

See [**?**, p 545] or Appendix B for common Laplace transforms, which are typically used in practice.

15 Approaches to problems

15.1 Specific mode excitation

These are typically stated in one of the following ways

1. "If there is no input and the initial condition vector is $\mathbf{x}(0) = \begin{bmatrix}3 & 1 & -6\end{bmatrix}$, will the state vector approach zero as time gets large?"
2. "If there is no input, find an initial condition vector such that the state vector approaches zero as time gets large."
3. "With initial condition vector $\mathbf{x}(0) = \begin{bmatrix}3 & 1 & -6\end{bmatrix}$ (or zero), choose a $\mathbf{B}$ matrix such that (or with a given $\mathbf{B}$, will) the state vector approach zero as time gets large."
4. There may be some variation of these with outputs instead of states.
5. There may be some variation of these with states or output going to zero in a finite time.

15.1.1 Approaches

There are four main ideas at work here: modal decomposition, stability, controllability, and observability. The main approach here follows.

1. Find the eigenvalues of $\mathbf{A}$. Typically at least one is unstable for these problems and they are distinct.
2. Find the eigenvectors of $\mathbf{A}$.
3. Depending on what is required, fully or partially construct the modal decomposition system with modal state variable $\mathbf{q}$.
4. An initial state in the modal state variable $\mathbf{q}(0)$ will not excite the unstable mode if it has no component in the direction of the unstable mode, since the homogeneous response can be written as (103),

$$\begin{aligned}\mathbf{x}_h(t) &= \mathbf{M}e^{\mathbf{\Lambda} t}\boldsymbol{\alpha} \\ &= \alpha_1\mathbf{m}_1e^{\lambda_1 t} + \alpha_2\mathbf{m}_2e^{\lambda_2 t} + ... + \alpha_n\mathbf{m}_ne^{\lambda_n t},\end{aligned}$$

and the initial conditions being applied determine the values of α_i, and $\alpha_i = 0$ will correspond to an eigenvector that has no component in the initial condition:

$$\mathbf{x}(0) = \alpha_1\mathbf{m}_1 + \alpha_2\mathbf{m}_2 + ... + \alpha_n\mathbf{m}_n.$$

The best way to think about it is this: we want α_j corresponding to the unstable (j-th) eigenvector to be zero.

Let's simply solve for $\mathbf{x}(0)$ in terms of $\boldsymbol{\alpha}$, make the j-component of $\boldsymbol{\alpha}$, the see what $\mathbf{x}(0)$ satisfies the equation. We have

$$\begin{aligned}\mathbf{x}(0) &= \mathbf{M}e^{\mathbf{\Lambda} 0}\boldsymbol{\alpha} \\ &= \mathbf{M}\mathbf{I}\boldsymbol{\alpha} \\ &= \mathbf{M}\boldsymbol{\alpha} \\ &= \alpha_1\mathbf{m}_1 + \alpha_2\mathbf{m}_2 + ... + \alpha_n\mathbf{m}_n\end{aligned}$$

So we choose $\boldsymbol{\alpha}$ as anything with $\alpha_j = 0$ (any and all unstable modes), then multiply by $\mathbf{M}$ to find $\mathbf{x}(0)$. Something interesting to note here is that

$$\begin{aligned}\boldsymbol{\alpha} &= \mathbf{M}^{-1}\mathbf{x}(0) \\ &= \mathbf{M}^{-1}\mathbf{M}\mathbf{q}(0) \\ &= \mathbf{q}(0)\end{aligned}$$

This shows the connection between the original and decoupled systems. The decoupled system has the identity matrix as its eigenvectors (and are therefore orthogonal)

(206)

Note that the eigenvectors are not necessarily orthogonal to each other (only if $\mathbf{A} = \mathbf{A}^\intercal$).

5. Controllability and observability criteria (especially 1) are useful when there is a nonzero input or the output is of interest. Specifically, in the modal decomposition state model we can easily determine if a mode is being excited by the input and if it is being observed in the output.

References

[1] Derek Rowell and David N. Wormley. *System Dynamics: An Introduction*. Prentice Hall, 1997.

[2] William L Brogan. *Modern Control Theory*. Prentice Hall, third edition, 1991.

A Impendance-based modeling

Notes on Generalized Impedances by J. L. Garbini

Generalized impedances are an extension of the concept of electrical impedances to systems of other domains. The table below lists the corresponding driving-point impedance definitions for five different energy modalities.

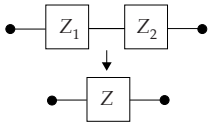
		Mechanical Translational	Mechanical Rotational	Electrical	Fluid	Thermal
Across Variable		v, velocity	ω, angular velocity	v, voltage	p, pressure	T, temperature
Through Variable		f, force	T, torque	i, current	q, volumetric flow	q, heat flow rate
Impedance $Z(s)$ **Admittance** $Y(s) = \frac{1}{Z(s)}$		$Z(s) = \frac{V(s)}{F(s)}$	$Z(s) = \frac{\Omega(s)}{T(s)}$	$Z(s) = \frac{V(s)}{I(s)}$	$Z(s) = \frac{P(s)}{Q(s)}$	$Z(s) = \frac{T(s)}{Q(s)}$
Impedance $Z(s)$	**A-Type**	mass, M: $\frac{1}{Ms}$	inertia, J: $\frac{1}{Js}$	capacitor, C $\frac{1}{Cs}$	fluid capacitor, C $\frac{1}{Cs}$	thermal capacitor, C $\frac{1}{Cs}$
	D-Type	damper, B $\frac{1}{B}$	r. damper, B $\frac{1}{B}$	resistor, R R	fluid resistor, R R	thermal resistor, R R
	T-Type	spring, K $\frac{s}{K}$	r. spring, K_r $\frac{s}{K_r}$	inductor, L Ls	fluid inductor, L Ls	—

Series and parallel combinations of impedances and admittances can be combined. In the following V and F represent the across and through variables respectively of any physical domain.

Series Combination

Elements sharing a common *through* variable are in *series*.

The *impedance* of elements connected in *series* is the sum of the individual *impedances*.

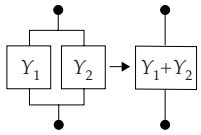


$$Z_1 = Z_1 + Z_2$$

Parallel Combination

Elements sharing a common *across* variable are in *parallel*.

The *admittance* of elements connected in *parallel* is the sum of the individual *admittances*.



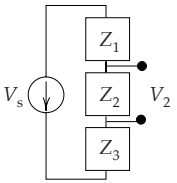
$$Y = Y_1 + Y_2$$

$$Z = \frac{1}{Y} = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

Simple transfer functions can be determined from impedance/admittance properties.

Across Variable Divider

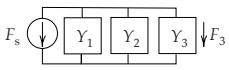
The complex amplitude of the *across variable* across a set of elements in *series* is divided among the elements in proportion their *impedances*.



$$T(s) = \frac{V_2(s)}{V_s(s)} = \frac{Z_2}{Z_1 + Z_2 + Z_3}$$

Through Variable Divider

The complex amplitude of the *through variable* through a set of elements in *parallel* is divided among the elements in proportion their *admittances*.



$$T(s) = \frac{F_2(s)}{F_s(s)} = \frac{Y_2}{Y_1 + Y_2 + Y_3}$$

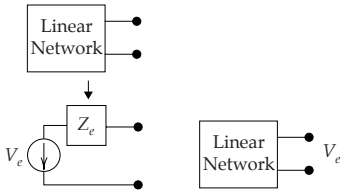
Thevenin and Norton equivalent networks are useful deriving transfer functions and in modeling systems that have a defined load impedance.

Thevenin's Theorem

A linear two-terminal network is equivalent to an across variable source V_e in *series* with an equivalent impedance Z_e, where

Z_e = the impedance of the network with all sources set equal to zero, and

V_e = an *across variable source* equal to the across variable that would appear across the *open* circuit terminals of the network.

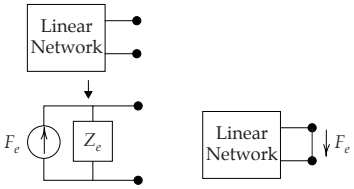


Norton's Theorem

A linear two-terminal network is equivalent to a through variable source F_e in *parallel* with an equivalent impedance Z_e, where

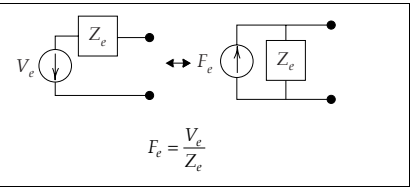
Z_e = the impedance of the network with all sources set equal to zero, and

F_e = a *through variable source* equal to the through variable that would flow through the *short* circuited terminals of the network.



Source Transformations

Since any linear two-terminal networks can be represented by either a Thevenin equivalent or a Norton equivalent, the two representations must be equivalent to each other.



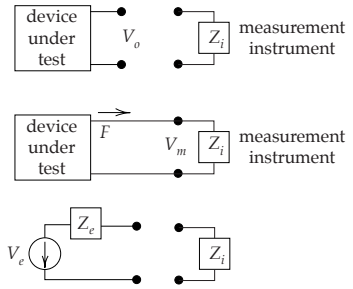
Measurement Loading

Across Variable Measurements

Suppose that we wish to measure an across variable at the output of a "device under test" with a "measurement instrument." The measurement instrument is attached across the terminals of interest. Of course we desired that the measured variable be undisturbed by the connection of the instrument. That is, we want V_m to be as nearly equal to V_o as possible. We say that the measurement instrument should not "load" the device under test.

The *output impedance* of the device under test is the equivalent impedance defined by its Thevenin model $Z_o = Z_e$ for the unloaded output terminals.

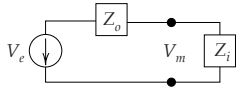
Similarly, the *input impedance* Z_i of the measurement instrument is the Thevenin equivalent impedance defined for its input terminals.



Connecting the Thevenin model for the device under test to the input impedance of the measurement instrument we have the network at the right.

The Thevenin equivalent across variable source is by definition equal to V_o, the value that we wish to measure. Applying the across variable divider rule: $\frac{V_m(s)}{V_o(s)} = \frac{1}{1+Z_o/Z_i}$.

Since we desire that the ratio approach unity, the input impedance of the measurement instrument must be large in comparison with the output impedance of the device under test: $Z_i >> Z_o$

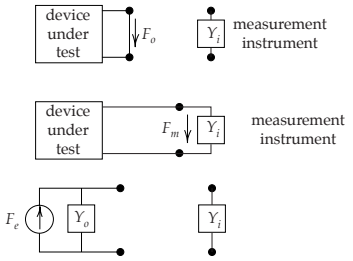


Through Variable Measurements

Alternately, suppose that we wish to measure a through variable in a device under test with a measurement instrument. In this case, the variable of interest flows through the measurement instrument. We desired that the measured variable be undisturbed by the connection of the instrument. That is, we want F_m to be as nearly equal to F_o as possible.

The *output admittance* of the device under test is the equivalent admittance defined by its Norton's model $Y_o = 1/Z_e$ for the unloaded output terminals.

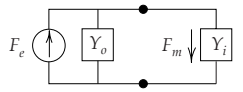
Similarly, the *input admittance* Y_i of the measurement instrument is the Norton equivalent admittance defined for its input terminals.



Connecting the Norton model for the device under test to the input admittance of the measurement instrument we have the network at the right.

The Norton equivalent through variable source is by definition equal to F_o, the value that we wish to measure. Applying the through variable divider rule: $\frac{F_m(s)}{F_o(s)} = \frac{1}{1+Y_o/Y_i}$.

Since we desire that the ratio approach unity, the input admittance of the measurement instrument must be large in comparison with the output admittance of the device under test: $Y_i >> Y_o$



B Laplace transforms

B.1 Laplace transform properties

Formula	Name, Comments	Sec.
$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$ $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Definition of Transform Inverse Transform	6.1
$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	Linearity	6.1
$\mathcal{L}\{e^{at} f(t)\} = F(s - a)$ $\mathcal{L}^{-1}\{F(s - a)\} = e^{at} f(t)$	s-Shifting (First Shifting Theorem)	6.1
$\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$ $\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{(n-1)}f(0) - \cdots$ $\cdots - f^{(n-1)}(0)$ $\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s}\mathcal{L}(f)$	Differentiation of Function Integration of Function	6.2
$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau$ $= \int_0^t f(t - \tau) g(\tau)\, d\tau$ $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$	Convolution	6.5
$\mathcal{L}\{f(t - a)\, u(t - a)\} = e^{-as} F(s)$ $\mathcal{L}^{-1}\{e^{-as} F(s)\} = f(t - a)\, u(t - a)$	t-Shifting (Second Shifting Theorem)	6.3
$\mathcal{L}\{tf(t)\} = -F'(s)$ $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\tilde{s})\, d\tilde{s}$	Differentiation of Transform Integration of Transform	6.6
$\mathcal{L}(f) = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$	f Periodic with Period p	6.4 Project 16

B.2 Laplace transform pairs

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
1	$1/s$	1	6.1
2	$1/s^2$	t	
3	$1/s^n \quad (n = 1, 2, \cdots)$	$t^{n-1}/(n-1)!$	
4	$1/\sqrt{s}$	$1/\sqrt{\pi t}$	
5	$1/s^{3/2}$	$2\sqrt{t/\pi}$	
6	$1/s^a \quad (a > 0)$	$t^{a-1}/\Gamma(a)$	
7	$\frac{1}{s-a}$	e^{at}	6.1
8	$\frac{1}{(s-a)^2}$	te^{at}	
9	$\frac{1}{(s-a)^n} \quad (n = 1, 2, \cdots)$	$\frac{1}{(n-1)!}\,t^{n-1}e^{at}$	
10	$\frac{1}{(s-a)^k} \quad (k > 0)$	$\frac{1}{\Gamma(k)}\,t^{k-1}e^{at}$	
11	$\frac{1}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)}\,(e^{at} - e^{bt})$	
12	$\frac{s}{(s-a)(s-b)} \quad (a \neq b)$	$\frac{1}{(a-b)}\,(ae^{at} - be^{bt})$	
13	$\frac{1}{s^2+\omega^2}$	$\frac{1}{\omega}\sin \omega t$	6.1
14	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	
15	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$	
16	$\frac{s}{s^2-a^2}$	$\cosh at$	
17	$\frac{1}{(s-a)^2+\omega^2}$	$\frac{1}{\omega}\,e^{at}\sin \omega t$	
18	$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos \omega t$	
19	$\frac{1}{s(s^2+\omega^2)}$	$\frac{1}{\omega^2}\,(1 - \cos \omega t)$	6.2
20	$\frac{1}{s^2(s^2+\omega^2)}$	$\frac{1}{\omega^3}\,(\omega t - \sin \omega t)$	
21	$\frac{1}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega^3}\,(\sin \omega t - \omega t \cos \omega t)$	6.6

	$F(s) = \mathcal{L}\{f(t)\}$	$f(t)$	Sec.
22	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t}{2\omega}\sin \omega t$	6.6
23	$\frac{s^2}{(s^2+\omega^2)^2}$	$\frac{1}{2\omega}\,(\sin \omega t + \omega t \cos \omega t)$	
24	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2 \neq b^2)$	$\frac{1}{b^2-a^2}\,(\cos at - \cos bt)$	
25	$\frac{1}{s^4+4k^4}$	$\frac{1}{4k^3}\,(\sin kt \cosh kt - \cos kt \sinh kt)$	
26	$\frac{s}{s^4+4k^4}$	$\frac{1}{2k^2}\sin kt \sinh kt$	
27	$\frac{1}{s^4-k^4}$	$\frac{1}{2k^3}\,(\sinh kt - \sin kt)$	
28	$\frac{s}{s^4-k^4}$	$\frac{1}{2k^2}\,(\cosh kt - \cos kt)$	
29	$\sqrt{s-a} - \sqrt{s-b}$	$\frac{1}{2\sqrt{\pi t^3}}\,(e^{bt} - e^{at})$	
30	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-(a+b)t/2} I_0\left(\frac{a-b}{2}\,t\right)$	5.6
31	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	5.5
32	$\frac{s}{(s-a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}}\,e^{at}(1 + 2at)$	
33	$\frac{1}{(s^2-a^2)^k} \quad (k > 0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-1/2} I_{k-1/2}(at)$	5.6
34	e^{-as}/s	$u(t-a)$	6.3
35	e^{-as}	$\delta(t-a)$	6.4
36	$\frac{1}{s}\,e^{-k/s}$	$J_0(2\sqrt{kt})$	5.5
37	$\frac{1}{\sqrt{s}}\,e^{-k/s}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
38	$\frac{1}{s^{3/2}}\,e^{k/s}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
39	$e^{-k\sqrt{s}} \quad (k > 0)$	$\frac{k}{2\sqrt{\pi t^3}}\,e^{-k^2/4t}$	
40	$\frac{1}{s}\ln s$	$-\ln t - \gamma \quad (\gamma \approx 0.5772)$	5.6
41	$\ln \frac{s-a}{s-b}$	$\frac{1}{t}\,(e^{bt} - e^{at})$	
42	$\ln \frac{s^2+\omega^2}{s^2}$	$\frac{2}{t}\,(1 - \cos \omega t)$	6.6
43	$\ln \frac{s^2-a^2}{s^2}$	$\frac{2}{t}\,(1 - \cosh at)$	
44	$\arctan \frac{\omega}{s}$	$\frac{1}{t}\sin \omega t$	
45	$\frac{1}{s}\,\mathrm{arccot}\, s$	$\mathrm{Si}(t)$	App. A3.1

C Fourier transforms

C.1 Fourier transform properties

The functions are periodic with period T, $a > 0$; and b, t_o, and $\omega_o = 2\pi/T$ are real constants with $n = 1, 2, \cdots$.

$f(t)$	$F(\omega)$
$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
$f(-t)$	$F(-\omega)$
$f(t - t_0)$	$F(\omega)\, e^{-j\omega t_0}$
$f(t)\, e^{j\omega_0 t}$	$F(\omega - \omega_0)$
$f(t) \cos \omega_0 t$	$\frac{1}{2} F(\omega - \omega_0) + \frac{1}{2} F(\omega + \omega_0)$
$f(t) \sin \omega_0 t$	$\frac{1}{2j} F(\omega - \omega_0) - \frac{1}{2j} F(\omega + \omega_0)$
$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$	$R(\omega)$
$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$	$jX(\omega)$
$f(t) = f_e(t) + f_o(t)$	$F(\omega) = R(\omega) + jX(\omega)$
$F(t)$	$2\pi f(-\omega)$
$f'(t)$	$j\omega F(\omega)$
$f^{(n)}(t)$	$(j\omega)^n F(\omega)$
$\int_{-\infty}^{t} f(x)\, dx$	$\frac{1}{j\omega} F(\omega) + \pi F(0)\, \delta(\omega)$
$-jt f(t)$	$F'(\omega)$
$(-jt)^n f(t)$	$F^{(n)}(\omega)$
$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t - x)\, dx$	$F_1(\omega) F_2(\omega)$

C.2 Fourier transform pairs

$f(t)$	$F(\omega)$
$f_1(t)\, f_2(t)$	$\frac{1}{2\pi} F_1(\omega) * F_2(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(y) F_2(\omega - y)\, dy$
$e^{-at} u(t)$	$\frac{1}{j\omega + a}$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2 + \omega^2}$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}\, e^{-\omega^2/(4a)}$
$p_a(t) = \begin{cases} 1 & \text{for } \lvert t\rvert < a/2 \\ 0 & \text{for } \lvert t\rvert > a/2 \end{cases}$	$a \frac{\sin\left(\frac{\omega a}{2}\right)}{\left(\frac{\omega a}{2}\right)}$
$\frac{\sin at}{\pi t}$	$p_{2a}(\omega)$
$te^{-at} u(t)$	$\frac{1}{(j\omega + a)^2}$
$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$	$\frac{1}{(j\omega + a)^n}$
$e^{-at} \sin bt\, u(t)$	$\frac{b}{(j\omega + a)^2 + b^2}$
$e^{-at} \cos bt\, u(t)$	$\frac{j\omega + a}{(j\omega + a)^2 + b^2}$
$\frac{1}{a^2 + t^2}$	$\frac{\pi}{a} e^{-a\lvert\omega\rvert}$
$\frac{\cos bt}{a^2 + t^2}$	$\frac{\pi}{2a}\left[e^{-a\lvert\omega - b\rvert} + e^{-a\lvert\omega + b\rvert}\right]$
$\frac{\sin bt}{a^2 + b^2}$	$\frac{\pi}{2aj}\left[e^{-a\lvert\omega - b\rvert} - e^{-a\lvert\omega + b\rvert}\right]$
$\delta(t)$	1
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\delta'(t)$	$j\omega$
$\delta^{(n)}(t)$	$(j\omega)^n$
$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$
$u(t - t_0)$	$\pi\delta(\omega) + \frac{1}{j\omega} e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
t	$2\pi j\, \delta'(\omega)$
t^n	$2\pi j^n \delta^{(n)}(\omega)$

$f(t)$	$F(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\sin\omega_0 t\,u(t)$	$\frac{\omega_0}{\omega_0^2-\omega^2}+\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\cos\omega_0 t\,u(t)$	$\frac{j\omega}{\omega_0^2-\omega^2}+\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$t\,u(t)$	$j\pi\,\delta'(\omega)-\frac{1}{\omega^2}$
$\frac{1}{t}$	$\pi j-2\pi j\,u(\omega)$
$\frac{1}{t^n}$	$\frac{(-j\omega)^{n-1}}{(n-1)!}[\pi j-2\pi j\,u(\omega)]$
$\operatorname{sgn} t$	$\frac{2}{j\omega}$
$\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\delta_{\omega_0}(\omega)=\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$

Other properties:

$$\int_{-\infty}^{\infty} f_1(t)\,f_2(t)\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(\omega)\,F_2^*(\omega)\,d\omega,$$

$$\int_{-\infty}^{\infty}|f(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\omega)|^2\,d\omega,$$

$$\int_{-\infty}^{\infty}f(x)\,G(x)\,dx=\int_{-\infty}^{\infty}F(x)\,g(x)\,dx.$$

D Periodic input response

System Response to Periodic Inputs