06.10 rldesign.multd Multiple derivative compensators

Lec. 06.6 rldesign.PD shows how to design a derivative compensator such that the compensated root locus of a control system can be made to include some test point $\psi \in \mathbb{C}$ where the designer would like a closed-loop pole (typically to satisfy transient response requirements). This derivative compensator has the form

$$C_D = K(s - z_c), \tag{1}$$

for gain $K \in \mathbb{R}$ and zero $z_c \in \mathbb{R}$. The crux of the design procedure is to compute via the root locus phase criterion[11] the *required* compensator phase contribution:

$$\theta_c = \pi - \angle GH(\psi) \tag{2}$$

for open-loop transfer function $GH(s)$. A trigonometric analysis shows that, for $\theta_c \in [-\pi, \pi]$, the compensator zero must be

$$z_c = \mathrm{Re}(\psi) - \mathrm{Im}(\psi)/\tan\theta_c. \tag{3}$$

The obvious limitation here is that if the required compensation θ_c is beyond $\pm\pi$, the derivative compensator of Eq. 1 cannot contribute sufficient phase. The strategy we adopt here is to augment the derivative compensator to include as many (equal) zeros as we need:

$$C_m = K(s - z_m)^m, \tag{4}$$

where z_m is a zero of multiplicity m. We call this a **multiple derivative compensator** or m-*derivative compensator*.

How do we select the compensator zero z_m and multiplicity m for a given θ_c? First, we determine m by determining how many π (or $-\pi$) contributions are required:[12,13]

11. The phase criterion was defined in Lec. 05.1 rlocus.def, Eq. 6.

Algorithm multd.1 the multiple derivative compensator algorithm.

function d_comp_m(ψ, $GH(s)$)
- $\theta_c \leftarrow \pi - \angle GH(\psi)$ ▷ *required phase comp*
- $m \leftarrow \text{ceiling}(\theta_c/\pi)$ ▷ *zeros needed*
- $\theta_m \leftarrow \theta_c/m$ ▷ *divide contributions*
- $z_m \leftarrow \mathrm{Re}(\psi) - \mathrm{Im}(\psi)/\tan\theta_m$ ▷ *trig*
- $C'_m \leftarrow (s - z_m)^m$ ▷ *comp sans gain*
- $K_m \leftarrow |C'_m(\psi)GH(\psi)|^{-1}$ ▷ *angle criterion*
- $C_m \leftarrow K_m C'_m$ ▷ *comp with gain*
- **return** C_m

end function

12. The function $\lceil\cdot\rceil$ is called the ceiling function and rounds up to the nearest integer.

13. Note that if $\theta_c \in [-\pi, \pi]$, the multiplicity $m = 1$ and the compensator is a regular derivative compensator.

$$m = \left\lceil \frac{|\theta_c|}{\pi} \right\rceil. \qquad (5)$$

With this, we can divide-up the the required phase contribution θ_c among the m zeros:

$$\theta_m = \theta_c/m. \qquad (6)$$

By construction, $\theta_m \in [-\pi, \pi]$, so the compensator zeros should be located at

$$z_m = \text{Re}(\psi) - \text{Im}(\psi)/\tan\theta_m. \qquad (7)$$

This is summarized in Algorithm multd.1.

Causality

A complication can arise when derivative compensation yields a closed-loop transfer function with more zeros than poles—a type of system called **non-causal** (non-non-causal systems are called **causal**). Non-causal systems are those that depend on *future* states, something classically[14] impossible to instantiate in real-time, and therefore a controller that creates such a control system is of no practical use.[15] Adding multiple zeros to a controller can easily yield such undesirable systems.
To mitigate this, we can include ι pure integrators $1/s$ into the compensator. They will obviously affect the root locus, so their effects must be taken into account during the zero compensator calculations. This is done by treating the open-loop transfer function as if it already had the compensator integrators $1/s^\iota$. Algorithm multd.2 summarizes this approach.

14. It gets complicated when considering relativity and quantum mechanics, which we do not, here.

15. Non-causal system models are useful for digital signal post-processing, but these are always *a posteriori*—i.e. “future” time is known because it is in the analytic past. Controllers do not have this luxury.

Algorithm multd.2 the multiple derivative compensator algorithm with ι integrators.

function d_comp_mi(ψ, $GH(s)$, ι)
 $\theta_c \leftarrow \pi - \angle GH(\psi)/s^\iota$ ▷ *required phase comp*
 $m \leftarrow \text{ceiling}(\theta_c/\pi)$ ▷ *zeros needed*
 $\theta_m \leftarrow \theta_c/m$ ▷ *divide contributions*
 $z_m \leftarrow \text{Re}(\psi) - \text{Im}(\psi)/\tan\theta_m$ ▷ *trig*
 $C'_m \leftarrow (s - z_m)^m/s^\iota$ ▷ *comp sans gain*
 $K_m \leftarrow |C'_m(\psi)GH(\psi)|^{-1}$ ▷ *angle criterion*
 $C_m \leftarrow K_m C'_m$ ▷ *comp with gain*
 return C_m
end function

Example 06.10 rldesign.multd-1

Design a controller to meet the