# 07.8 freq.freqtime Relations among time and frequency domain reps

## The second-order assumption

As in root locus design, our transient response characteristics—such as percent overshoot $\%OS$, settling time $T_s$, and peak time $T_p$—can be related exactly to second-order response characteristics $\zeta$ and $\omega_n$, which have their own interpretations in the frequency domain and are related to key features of the Bode plot. This often gets us close enough that small iterations on the initial design can achieve the desired transient response.
The second-order approximation assumes an open-loop transfer function of the form

$$G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)} \tag{1}$$

which yields a closed-loop transfer function

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \tag{2}$$

which has a familiar frequency response.

*Bandwidth*

The term **bandwidth** appears in many contexts, but in control theory when using the second-order assumption it has a very specific definition.

**Definition 07 freq.1: bandwidth**

et a system have a transfer function $G(s)$ and frequency response function $G(j\omega)$. The bandwidth $\omega_{BW}$ of the system is the angular frequency at which $|G(j\omega)|$ is 3 dB less than $|G(j0)|$.

## Closed-loop percent overshoot from the closed-loop bandwidth

It is straightforward to show that the bandwidth of the second-order closed-loop transfer of Eq. 2 is related to its natural frequency $\omega_n$ and damping ratio $\zeta$ by the expression

$$\omega_{BW} = \omega_n \left( (1 - 2\zeta^2) + (4\zeta^4 - 4\zeta^2 + 2)^{1/2} \right)^{1/2}. \tag{3}$$

So if a system behaves approximately like this second-order system, Eq. 3 relates the closed-loop frequency response characteristic $\omega_{BW}$ and the closed-loop time response characteristics $\omega_n$ and $\zeta$. This is a big step, but we often design the speed of response in terms of settling time $T_s$, peak time $T_p$, and rise time $T_r$. We already have relationships for these quantities and $\omega_n$ and $\zeta$, the consequences of two of which when applied to Eq. 3 are shown below:

$$\omega_{BW} = \frac{4}{T_s \zeta} \left( (1 - 2\zeta^2) + (4\zeta^4 - 4\zeta^2 + 2)^{1/2} \right)^{1/2} \tag{4a}$$

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1 - \zeta^2}} \left( (1 - 2\zeta^2) + (4\zeta^4 - 4\zeta^2 + 2)^{1/2} \right)^{1/2}. \tag{4b}$$

Furthermore, it can be shown that, when it exists (which it does for $0 < \zeta < 1/\sqrt{2}$), the **peak magnitude** $M_p = \max_\omega |H(j\omega)|$ is

$$M_p = \frac{1}{2\zeta\sqrt{1 - \zeta^2}}. \tag{5}$$

Of course, percent overshoot $\%OS$ is directly related to $\zeta$ by the equations

$$\%OS = 100 \exp \frac{-\zeta\pi}{\sqrt{1 - \zeta^2}} \qquad \Longleftrightarrow \qquad \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}} \tag{6}$$

so $M_p$ can be directly related to $\%OS$.

## Closed-loop percent overshoot and damping ratio from the open-loop phase margin

From closed-loop considerations of the frequency response, we have learned to determine some closed-loop time response characteristics. Now we learn to determine one of these characteristics—percent overshoot $\%OS$—from *open*-loop frequency response.

From the transfer function of Eq. 1, it is straightforward to relate the phase margin $\Phi_M$ of the open-loop transfer function to the damping ratio $\zeta$ via the expressions

$$\Phi_M = \arctan \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1 + 4\zeta^4}}} \iff \tag{7}$$

$$\zeta = \frac{\tan \Phi_M}{2(1 + \tan^2 \Phi_M)^{1/4}}. \tag{8}$$

As we know from Eq. 6, percent overshoot $\%OS$ is directly related to $\zeta$, so $\Phi_M$ can be directly related to $\%OS$, as shown in Fig. freqtime.1.

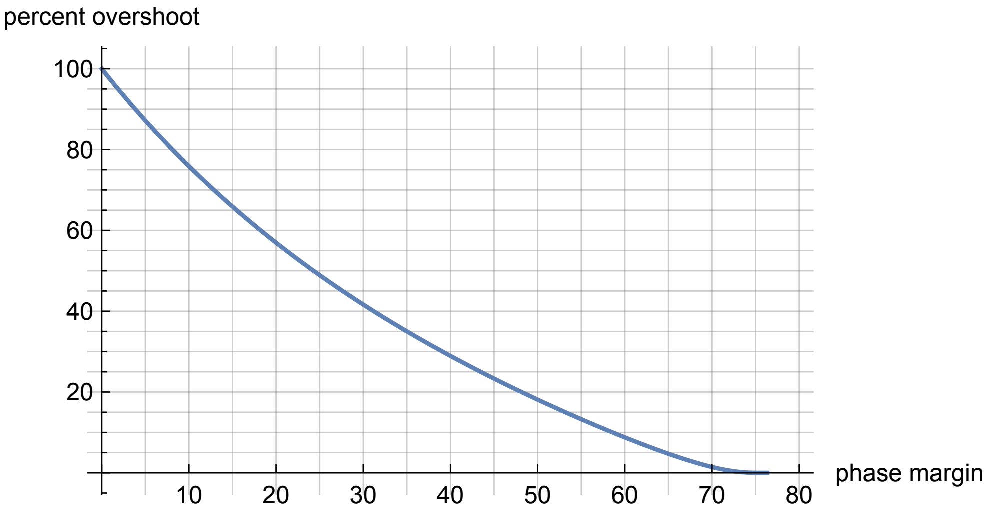


**Figure freqtime.1:** percent overshoot $\%OS$ versus phase margin $\Phi_M$.

## Closed-loop settling and peak times from the open-loop frequency response

We introduced the concept of bandwidth in above and related the closed-loop bandwidth $\omega_{BW}$ to settling time $T_s$ and peak time $T_p$ in Eq. 4. There is a method, which we present but leave underived, that allows us to find the closed-loop bandwidth of many systems from the open-loop frequency response, allowing us to relate the open-loop frequency response to $T_s$ and $T_p$. The method is based on the following insight: the closed-loop bandwidth is approximately equal to the frequency at which the magnitude of the open-loop frequency response is in the interval $[-6, -7.5]$ dB *if* the phase of the open-loop frequency response is in the interval $[-135, -225]$ deg. This gives us a method to approximate closed-loop $T_s$ and $T_p$ by inspecting the open-loop frequency response. Here's the method:

1. estimate the closed-loop bandwidth $\omega_{BW}$ by finding the frequency at which the magnitude of the open-loop frequency response is in the interval $[-6, -7.5]$ dB;
2. verify that open-loop phase at $\omega_{BW}$ is in the interval $[-135, -225]$ deg;
3. determine $\zeta$ via the phase margin (Eq. 7, Fig. freqtime.2); and
4. estimate $T_s$ and $T_p$ via Eq. 4.

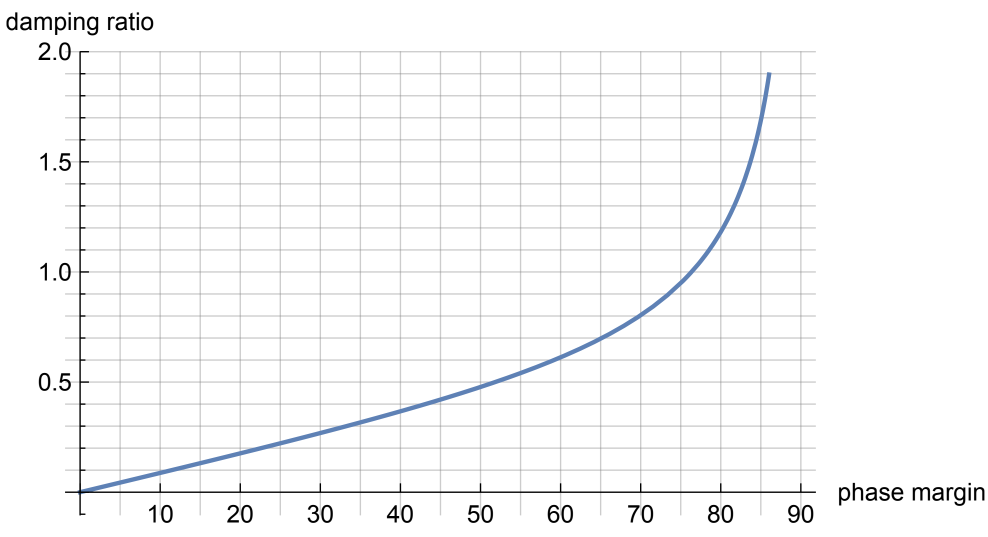


**Figure freqtime.2:** damping ratio $\zeta$ versus phase margin $\Phi_M$.