

## 09.1 ss.sfdbck Controller design method

We will consider single-input single-output (SISO) control plants that can be written with input  $u$ ; state vector  $x$ ; output  $y$ ; state model matrices  $A$ ,  $B$ ,  $C$ , and  $D$ ; and state and output equations

$$\dot{x} = Ax + Bu \tag{1a}$$

$$y = Cx + Du. \tag{1b}$$

Plants of this form can be written in block diagram form, as illustrated in Fig. sfdbck.1. In general, SISO systems are of order  $n$  with  $n$  state variables.

Let us consider the following feedback control method called **state feedback control**. We will feed back the state vector  $x$ , operate on it with a  $1 \times n$  vector of gains  $K \in \mathbb{R}^n$ , and subtract the result from the command  $r$ , the result of which becomes the input  $u$ , as shown in Fig. sfdbck.2.

The control problem for state feedback control is to determine the  $n$  gains in  $K$  such that the closed-loop poles are located in desirable positions. The gain  $N \in \mathbb{R}$  is provided for steady-state error considerations, which will be addressed in Lec. 09.1 ss.sfdbck. A new state model can be derived for the closed-loop system as follows. Let us consider the command  $r$  to be our new “input,” instead of  $u$ , which is now the control effort. From the block diagram,

$$u = Nr - Kx, \tag{2}$$

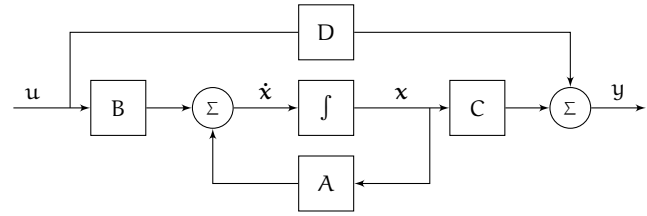
which can be substituted into Eq. 1 to define the new state model

$$\dot{x} = (A - BK)x + NBr \tag{3a}$$

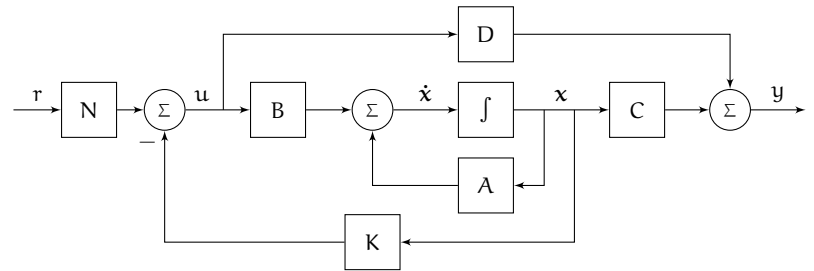
$$y = (C - DK)x + NDr. \tag{3b}$$

The eigenvalues of  $A - BK$ , which can be found from equating zero and the **closed-loop characteristic polynomial**

$$P_K = \det (sI - A + BK), \tag{4}$$



**Figure sfdbck.1:** the plant state model of Eq. 1 written in block diagram form.



**Figure sfdbck.2:** the state feedback control block diagram.

are equal to the closed-loop poles, which we would like to place in specific locations. Those specific locations can be specified by the **design characteristic polynomial**  $P_d$ .  $P_K$  depends on the  $n$  gains  $K_i$ , and  $n$  equations can be found by equating the polynomial coefficients of  $P_K$  and  $P_d$ .

Solving for  $K_i$  is straightforward but can be very tedious in the general case. Let the coefficients of  $P_d$  be  $\delta_i$  and those of  $P_K$  be denoted  $\kappa_i$ . Then the  $n \times 1$  vector containing  $\kappa_i$  can be expressed as a linear combination of  $K_i$  as

$$\kappa = \mathcal{K} \mathbf{K}^T, \quad (5)$$

where  $\mathcal{K}$  is an  $n \times n$  matrix of coefficients that were derived from  $A$  and  $B$ . Let  $\delta$  be the  $n \times 1$  vector of components  $\delta_i$ . Since the vector  $\delta$  is specified by our design requirements, we can solve for  $\mathbf{K}$  as follows.

$$\kappa = \delta, \quad (6)$$

and therefore,

$$\begin{aligned} \mathcal{K} \mathbf{K}^T = \delta &\implies \\ \mathbf{K}^T = \mathcal{K}^{-1} \delta &\implies \\ \mathbf{K} = (\mathcal{K}^{-1} \delta)^T. &\quad (7) \end{aligned}$$

Eq. 7 is valid for all cases in which  $\mathcal{K}$  is invertible.<sup>1</sup> However, there is a special form of the original state-space model that always yields a simple solution for  $\mathbf{K}$ : the **phase-variable canonical form** (see [Appendix B.02](#)).

1. We leave the following as an open question: under what conditions is  $\mathcal{K}$  invertible?

## Solving for the gain via the phase-variable canonical form

The phase-variable canonical form of the original system is:

$$\dot{\mathbf{x}}_c = A_c \mathbf{x}_c + B_c u \quad (8a)$$

$$y = C_c \mathbf{x}_c + D_c u \quad (8b)$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad (8c)$$

$$C_c = [c_1 \quad c_2 \quad \cdots \quad c_n], \quad \text{and} \quad D_c = [d_1], \quad (8d)$$

where the components  $a_i$  are defined by the original characteristic polynomial

$$P = \det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0. \quad (9)$$

With  $A_c$  defined, the form of the feedback state model with feedback row vector  $K_c$  is:

$$A'_c = A_c - B_c K_c, \quad B'_c = B_c, \quad (10a)$$

$$C'_c = C_c - D_c K_c, \quad \text{and} \quad D'_c = D_c. \quad (10b)$$

$A'_c$  deserves further attention. The special canonical form of  $A_c$  and  $B_c$  makes the expression for  $A'_c$  simply

$$A'_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & & 0 \\ 0 & 0 & \cdots & 1 \\ -(a_0 + K'_1) & -(a_1 + K'_2) & \cdots & -(a_{n-1} + K'_n) \end{bmatrix}, \quad (11)$$

where  $K'_i$  is the row vector of gains in the phase-variable canonical basis. The design characteristic polynomial coefficients  $\delta_i$  must equal the characteristic polynomial coefficients

$$\delta_i = a_i + K'_{i+1}, \quad (12)$$

which gives

$$K'_i = \delta_{i-1} - a_{i-1}. \quad (13)$$

This yields  $K'$ . If we equate the feedback

$$\begin{aligned} \mathbf{K}\mathbf{x} &= \mathbf{K}'\mathbf{x}_c \implies \\ \mathbf{K} &= \mathbf{K}'\mathbf{T}_c. \end{aligned} \quad (14)$$

Let  $\mathcal{U}$  and  $\mathcal{U}_c$  be the controllability matrices for the original basis and the phase-variable canonical basis, respectively. From [Appendix B.02](#), we can compute the transformation matrix to be

$$\mathbf{T}_c = \mathcal{U}_c \mathcal{U}^{-1}. \quad (15)$$

### Steady-state error

We can use the gain  $N$  to drive the closed-loop steady-state error to zero for step inputs. The idea is that we can scale the input by the reciprocal of the closed-loop steady-state error. Let  $G_{CL}(s)$  be the closed-loop transfer function. From the final value theorem for a unit step input,

$$N = \lim_{s \rightarrow 0, N \rightarrow 1} 1/G_{CL}(s). \quad (16)$$

If  $N$  is nonzero and finite, the response will have zero steady-state error. Although it is derived from unit step inputs, we can apply this formula to slowly varying inputs as well.

### Example 09.1 ss.sfbck-1

### re: state feedback pole placement design

Given the state-space model

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix} & \mathbf{B} &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \mathbf{C} &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} & \mathbf{D} &= \begin{bmatrix} 0 \end{bmatrix}, \end{aligned}$$

- design a controller with 15% overshoot and a
- settling time of 1 sec.





