09.2 ss.exe Exercises for Chapter 09 ss

A

Mathematical topics

A.01 Complex functions

A **complex function** f maps a subset of the complex plane to the complex plane (i.e. $f : \mathbb{C} \to \mathbb{C}$). For instance, a complex function f can map a single complex number s_0 to another $s_1 = f(s_0)$.

A **curve** in the complex plane is defined as a continuous function mapping a closed interval of the reals to the complex plane. A **contour** is defined as a directed curve consisting of a finite set of directed smooth curves, the final endpoint of which is identical to the starting point (Fig. A.01.1 shows a plot of a contour Γ).

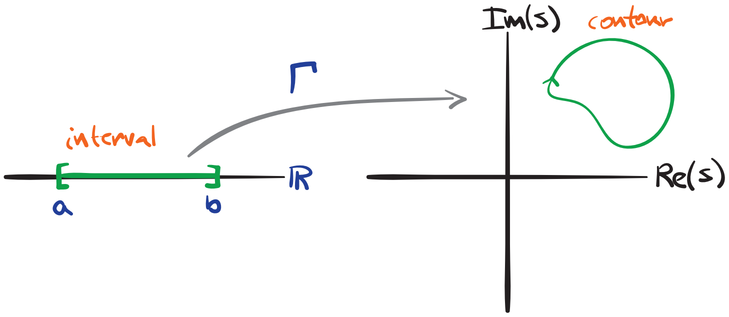

Figure A.01.1: illustrating the definition of a complex contour Γ.

A contour can be mapped by a complex function, and this is our primary concern. The image of a contour Γ mapped by a complex function f is itself a contour $f(\Gamma)$, as shown in Fig. A.01.2.

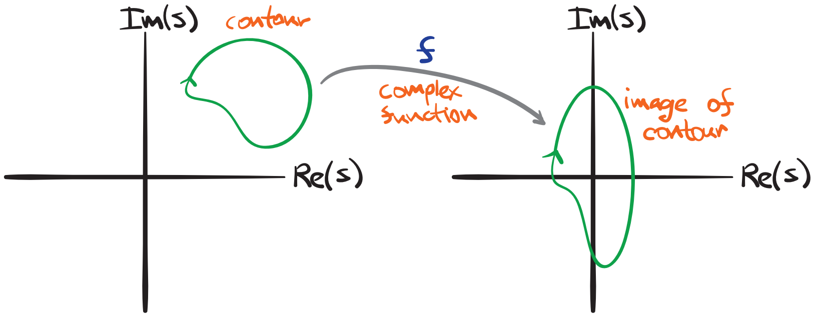

Figure A.01.2: a representation of a complex function f mapping a contour.

Complex functions are of interest in control theory because transfer functions, one of the central mathematical objects of control theory, are complex functions. The utility of evaluating and mapping contours with complex functions arises especially in root-locus design and frequency response design (especially for the Nyquist stability criterion).

Example A.01-1

re: transfer function mapping a single point

Map the complex point $s = 1 + j3$ with the transfer (complex) function

$$H(s) = \frac{s+4}{s-1}.$$

Sometimes we say that we are "evaluating" the transfer function at the point $s = 1 + j3$.

A geometric interpretation of complex functions

It is often helpful to interpret the complex mapping of a point or a contour geometrically. Let us consider a transfer (complex) function $H(s)$ with complex zeros z_i, complex poles p_j, and real scaling factor k. Considering each factored term of the transfer function in terms of its magnitude and phase, we can write the magnitude and phase of the transfer function as follows.

Equation 1 magnitude and phase of a transfer function

We can interpret this geometrically as follows. Let us consider the evaluation of Eq. 1 at a specific complex value ψ. The differences $\psi - z_i$ and $\psi - p_i$ can be thought of as vectors in the complex plane with tails at z_i and p_i and heads at ψ. Fig. A.01.3 shows this geometric interpretation with $p_{1,2} = -3 \pm j3$, $z_1 = 1$, and $\psi = 3 + j4$.

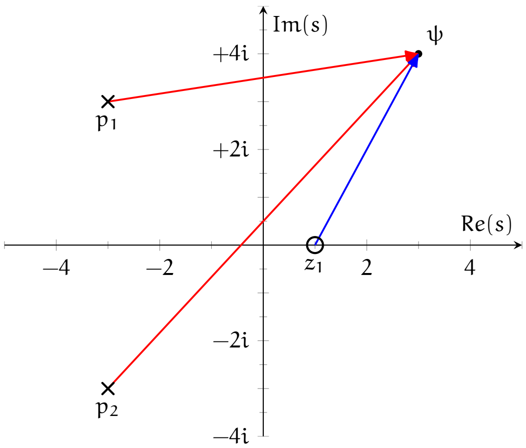

Figure A.01.3: an example of a geometric interpretation of the evaluation of a complex function with poles $p_{1,2}$ and zero z_1 at a complex value $s = \psi$.

Example A.01-2

re: transfer function mapping a contour

Let $I = [0, 2\pi]$. Let the contour $\Gamma : I \to \mathbb{C}$ be defined parametrically, with $t \in I$, as

$$\Gamma(t) = \sin t + j \cos t.$$

Map Γ with the transfer function

$$H(s) = \frac{s}{s^2 + 2s + 2}$$

and plot the result.

```
\[CapitalGamma][t_] := {Sin[t], Cos[t]}
H[s_] := (s + 1)/(s^2 + 2*s + 2);

ps = {Blue, Arrowheads[{0, .05, .05, .05}]};
mappingcontour = Animate[
  {
    ParametricPlot[
      \[CapitalGamma][t], {t, 0, T},
      PlotRange -> {-1, 1},
      PlotStyle -> ps,
      PlotLabel -> "\[CapitalGamma]"
    ] /.
      Line -> Arrow,
    ParametricPlot[
      H[Complex @@ \[CapitalGamma][t]] // {Re[#], Im[#]} &,
      {t, 0.001, T},
      PlotRange -> {-1.5, 1.5},
      PlotStyle -> ps,
      PlotLabel -> "H(\[CapitalGamma])"
    ] /.
        Line -> Arrow
  } // GraphicsRow,
  {T, 0, 2*\[Pi]}
]
```

Figure A.01.5: a basic Mathematica script for visualizing the transfer function mapping of Example A.01-2. A more thorough notebook is available here.

B

Linear systems theory topics

B.01 Controllability, observability, and stabilizability

The three topics controllability, observability, and stabilizability are three topics of central concern to linear systems theory.

Controllability

Controllability is defined as follows.

Definition B.1: controllable and uncontrollable

f there exists some input to a linear system such that any initial state in its state space can be evolved in finite time to any final state in its state space, the system is *controllable*. Otherwise, the system is *uncontrollable*.

A given system's controllability can be determined from the following.

Definition B.2: controllability matrix

et a linear system of order n and number of inputs r have state space $\{A, B, C, D\}$. We define the $n \times nr$ *controllability matrix* to be

$$\mathcal{U} = \left[\, B \,|\, AB \,|\, A^2B \,|\, \dots \,|\, A^{n-1}B \,\right].$$

The following well-known theorem, left unproven here, allows us to easily determine the controllability of a given system.

Theorem B.3: controllability

linear system is controllable if its controllability matrix has full rank. If it is less than full rank, the linear system is uncontrollable.

B.02 Canonical forms of the state model

There are several canonical forms for the state equations, all of which can be found via basis transformations from other forms.

Phase-variable canonical form

The **phase-variable canonical form** is represented by the SISO[1] state model

$$\dot{\mathbf{x}}_c = A_c \mathbf{x}_c + B_c u \quad (1a)$$

$$y = C_c \mathbf{x}_c + D_c u \quad (1b)$$

where

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad (1c)$$

$$C_c = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}, \text{ and} \quad D_c = \begin{bmatrix} d_1 \end{bmatrix}. \quad (1d)$$

1. There are phase-variable canonical forms for MIMO systems as well, but these are less standardized.

In order to transform a SISO system $\{A, B, C, D\}$ with state vector $\mathbf{x}$ to phase-variable canonical form, we change bases via the substitution of $\mathbf{x} = T_c \mathbf{x}_c$ into the original system, which gives

$$A_c = T_c^{-1} A T_c, \qquad B_c = T_c^{-1} B, \quad (2a)$$

$$C_c = C T_c, \text{ and} \qquad D_c = D. \quad (2b)$$

The special form of Equation 1 yields the following characteristic polynomial:

$$s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0. \quad (3)$$

Recall that eigenvalues of a system are invariant to basis change, and therefore so is its characteristic polynomial. From this we can conclude that A_c can be completely determined by finding the characteristic polynomial of the original matrix A. B_c is already fully determined, but C_c and D_c remain undetermined. They may be found by discovering the transformation matrix T_c and substituting it into Equation 2.

Finding the phase-variable canonical transformation

The phase-variable canonical transformation matrix T_c can be found by relating the controllability matrices of the original form and the canonical form.

Theorem B.4: phase-variable canonical transformation

he transformation matrix from a system representation with controllability matrix $\mathcal{U}$ to a phase-variable canonical transformation with controllability matrix $\mathcal{U}_c$ is

$$T_c = \mathcal{U}_c \mathcal{U}^{-1}. \tag{4}$$

By the Definition of the controllability matrix, the original controllability matrix is

$$\mathcal{U} = \left[\, B \,|\, AB \,|\, A^2 B \,|\, \ldots \,|\, A^{n-1} B \,\right] \tag{5}$$

and that of the canonical form is

$$\mathcal{U}_c = \left[\, B_c \,|\, A_c B_c \,|\, A_c^2 B_c \,|\, \ldots \,|\, A_c^{n-1} B_c \,\right]. \tag{6}$$

Note that $\mathcal{U}$ and $\mathcal{U}_c$ are both known from above. We relate the two forms by applying Equation 2 to Equation 6 to yield

$$\mathcal{U}_c = \left[\, T_c^{-1} B \,|\, T_c^{-1} AB \,|\, T_c^{-1} A^2 B \,|\, \ldots \,|\, T_c^{-1} A^{n-1} B \,\right] \tag{7a}$$

$$= T_c \mathcal{U}, \tag{7b}$$

to yield

$$T_c = \mathcal{U}_c \mathcal{U}^{-1}.$$

C

Physical topics

C.01 Decibels

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