stab.tf Stability from the transfer function

Stability from the poles of a closed-loop transfer function

From our definitions in terms of the free response (Lecture stab.intro), we see that a closed-loop LTI system is assymptotically **stable** if all its poles[5] have negative real parts (i.e. are in the left half-plane).

5. Recall that poles, eigenvalues, and roots of the characteristic equation are all equivalent.

Conversely, a closed-loop LTI system is **unstable** if it has at least one pole with a positive real part (i.e. in the right half-plane) and/or has poles of multiplicity greater than one on the imaginary axis.

Finally, a closed-loop LTI system is **marginally stable** if it is not unstable but has at least one pole with zero real part (i.e. on the imaginary axis) and if none of these has multiplicity greater than one.

Example stab.tf-1

re: Stability of a closed-loop transfer function from its poles

Given the plant transfer function

$$G(s) = \frac{1}{(s^2 + 3)}$$

find the unity (negative) feedback closed-loop transfer function and comment on its stability. Let the command be R(s) and the output Y(s).

Stability from the form of a closed-loop transfer function

Let $a_i, b_i, c \in \mathbb{R}$ be constant coefficients and the denominator of a closed-loop transfer function be the polynomial

$$b_n s^n + b_{n-1} s^{n-1} + \cdots + b_0 = c(s - a_1)(s - a_2)\cdots(s - a_n). \tag{1}$$

If a system is stable, it must have all left half-plane poles, so

1. all a_i must have negative real parts, which (non-obviously) implies that
2. all b_i must be positive and, additionally,
3. all b_i must be nonzero for $0 \leqslant i \leqslant n$ (i.e. no "missing" powers of s).

However, these b_i conditions are merely **necessary conditions** for stability, meaning that they are necessary for stability, but not **sufficient** (something more is needed to ensure stability).[6] However, if they are not met, this is a sufficient condition to draw the conclusion that the control system is unstable (i.e. nothing more needed).

necessary conditions

sufficient conditions

6. The logical statement $P \Rightarrow Q$ means P **is sufficient for** Q and Q **is necessary for** P. That is, if P then Q (sufficiency) and if not Q then not P (necessity). Necessity and sufficiency are duals. Let P be "the system is stable" and Q be "all b_i are positive." Then if any b_i is negative ($\neg Q$), then the system is unstable ($\neg P$). But if Q, it does not necessarily follow that P–more information is required.

Example stab.tf-2

re: Stability of a closed-loop transfer function by inspection

Given the closed-loop transfer functions

$$G_1(s) = \frac{s+4}{(s+3)(s+10)(s+22)}, \quad G_2(s) = \frac{s^2+2s+5}{s^2-5s+8},$$
$$G_3(s) = \frac{1}{s^3+s+4}, \text{ and} \quad G_4(s) = \frac{s^2+5}{s^4+3s^3+s^2+s+3}.$$

comment on the stability of each without solving for poles.