trans.exact Exact analytical trans response char of first- and second-order sys

First-order systems without zeros

A first-order system without zeros has a transient response characterized by a **time-constant** τ that appears in the general response as

time-constant

$$\_\, e^{-t/\tau} + \_\_\_. \tag{1}$$

The transient exponential decays such that in three time constants 3τ only 5 % of the term remains; in 5τ, less than 1 %.

There is neither peak nor overshoot for this type of response. However, the **rise time** for these systems is found by solving the time-domain differential equation

rise time

$$\tau \dot{y}(t) + y(t) = K u(t) \tag{2}$$

with output variable y, input variable u, and real constant K. It is easily shown that the solution to Eq. 2 in Eq. 2 is, for a unit step input,

$$y(t) = K\left(1 - e^{-t/\tau}\right), \tag{3}$$

from which we discover that the steady-state value is

$$\begin{aligned} y_{ss} &= \lim_{t\to\infty} y(t) && (4a) \\ &= K. && (4b) \end{aligned}$$

The rise time is, by definition, the duration of the time interval $[t_1, t_2]$ such that

$$\begin{aligned} y(t_1) &= 0.1 y_{ss} \text{ to} && (5a) \\ y(t_2) &= 0.9 y_{ss}. && (5b) \end{aligned}$$

The first of these yields

$$K\left(1 - e^{-t_1/\tau}\right) = 0.1K \Rightarrow \quad (6a)$$

$$t_1 = -\tau \ln 0.9 \quad (6b)$$

$$\approx 0.1054\tau. \quad (6c)$$

Solving in an analogous fashion, we find $t_2 \approx 2.3026\tau$. The interval, then, is $t_2 - t_1 = 2.1972\tau$.

settling time

natural frequency

damping ratio

Equation 7 first-order system rise time

Finally, the **settling time** can be derived in a fashion similar to the rise time.

Equation 8 first-order system settling time

Second-order systems without zeros

Second-order system transient responses are characterized by a **natural** (angular) **frequency** ω_n and **damping ratio** ζ. It is helpful to recall the complex-plane graphical representation of the pole-zero plot for a second-order system without zeros, as shown in Fig. exact.1.

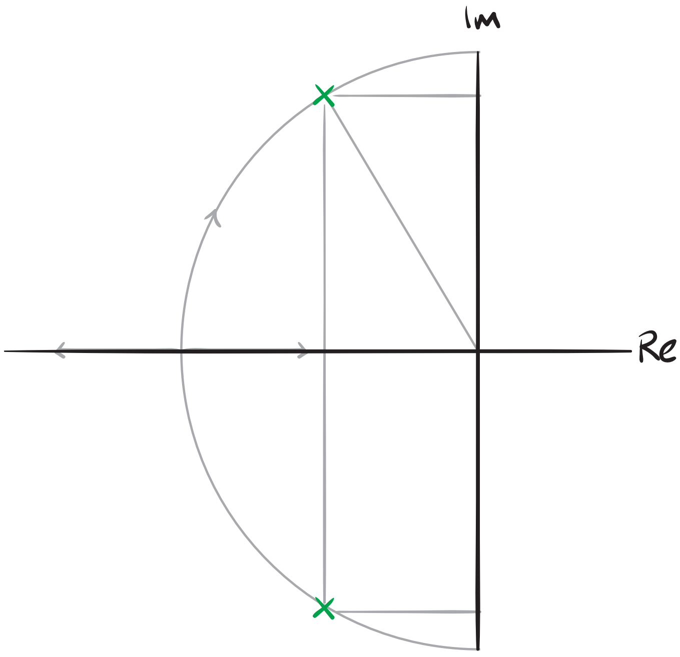

Figure exact.1: the relationship between the pole-zero plot of a second-order system with no zeros and ω_n and ζ.

Following a procedure very similar to that for first-order systems, the following relationships can be derived.

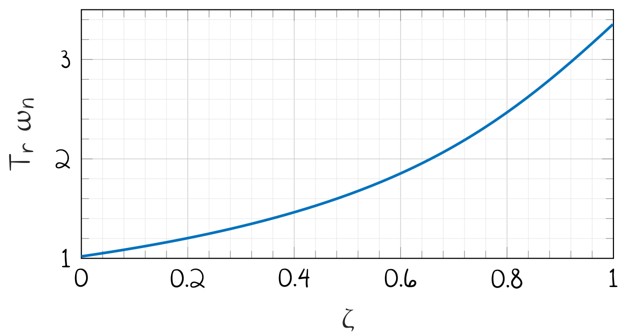

Figure exact.2: the relationship between rise time, natural frequency, and damping ratio.

rise time

The **rise time** T_r does not have an analytical solution in terms of ω_n and ζ. However, Fig. exact.2 shows numerical solutions for T_r scaled by ω_n for $\zeta \in (0, 1)$.

The **peak time** T_p has the following, simple expression

peak time

$$T_p = \frac{\pi}{\omega_d}, \tag{9}$$

where $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ is the **damped natural frequency**.

damped natural frequency

The **percent overshoot** %OS is related directly to ζ as follows

percent overshoot

$$\%OS = 100 \exp \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \quad \Leftrightarrow \tag{10}$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}. \tag{11}$$

Finally, the **settling time** T_s is expressed as

settling time

$$T_s = \frac{4}{\zeta\omega_n}. \tag{12}$$