rldesign.PD Proportional-derivative (PD) controller design

Thus far, our designs have been restricted to closed-loop pole locations on the original root locus. We could add integral or lag compensation for steady-state error performance and vary the gain for transient response performance. But what if we desire closed-loop poles $p_{1,2}$ to be in a location that the root locus does not intersect?

Among many possible methods to address this, we pursue the following: a **derivative compensator** with zero location z_c chosen such that the root locus intersects $p_{1,2}$, with form

derivative compensator

$$K(s - z_c), \tag{1}$$

where $K \in \mathbb{R}$ is a gain. This compensator is called "derivative" because its primary effect on the overall controller's operation on the error e is a new factor of s, yielding a scaling of the term $sE(s) = \dot{e}(t)$.

The effect of this zero is to pull the locus toward it. Consider the simple plant of Fig. PD.1. Suppose we would like to speed up the closed-loop response, but cannot because, no matter how much gain we use, the settling time is fixed by the vertical asymptotes. If we use a compensator zero at z_c, we can pull the locus leftward, as shown in Fig. PD.2. Varying z_c from $-\infty$ to 0, we see that any location left of -2 can be intersected. In fact, if we consider both positive and negative gains for this example, we can place a desired closed-loop pole at any location in the complex plane!

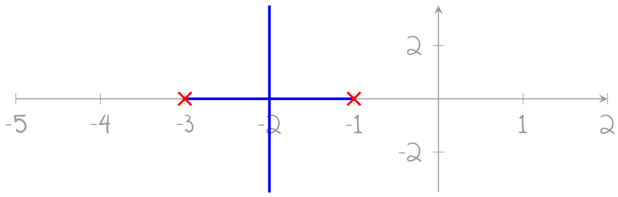

Figure PD.1: root locus for a simple plant with two poles.

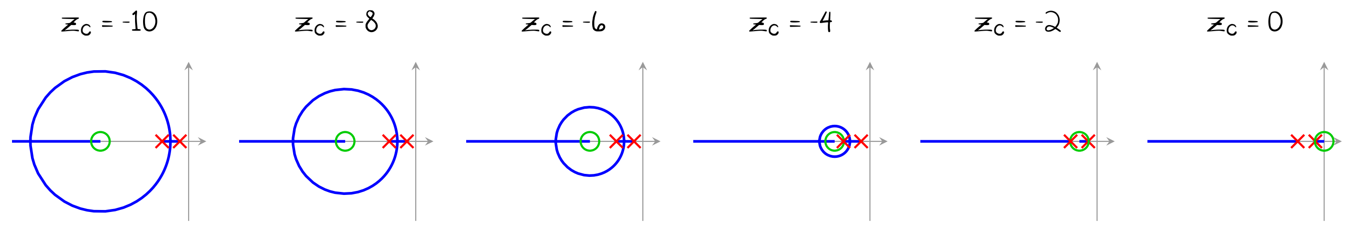

Figure PD.2: root locus (blue) for plant with poles (red) compensated with a zero (green) at z_c. Note that varying z_c yields root loci that can intersect any point in the complex plane if negative gains are considered. An animation corresponding to this figure can be found at https://youtu.be/VZbT_2bT2xU.

A way to approach designing a controller for a plant G with a derivative compensator C is to consider the compensator zero's effect on the phase criterion, which must always be satisfied at points on the root locus:

$$\angle(G(s)C(s)) = \pi. \qquad (2)$$

In order for a desired point $s = \psi$ to be on the root locus, then,[3]

3. The 2π modulo in these expressions is suppressed for clarity.

Let this angle $\angle(\psi - z_c)$, called the **compensator angle**, be given the symbol

compensator angle

$$\theta_c \equiv \angle(\psi - z_c). \qquad (3)$$

Then

$$z_c = \text{Re}(\psi) - \text{Im}(\psi)/\tan\theta_c \quad (\theta_c \in [-\pi, \pi]), \qquad (4)$$

where we have limited the application of this result to $\theta_c \in [-\pi, \pi]$ because a single zero can contribute angles in this interval only.[4,5] This result is to be used in the design procedure that follows. It can be understood geometrically as the position of z_c such that the angle of the vector with tail at z_c and head at ψ is θ_c.

4. See Lec. rldesign.multd for how to handle required angle compensations beyond $\pm\pi$.

5. Note that $\theta_c \in [-\pi, 0)$ is possible only when $\text{Im}\,\psi < 0$ and $\theta_c \in (0, \pi]$ is possible only when $\text{Im}\,\psi > 0$.

Design procedure

The following procedure provides a starting-point for proportional-derivative controller design. Let's assume the transient response specification is such that we desire a closed-loop pole to be located at $s = \psi$.

1. Design a proportional controller to meet transient response requirements by choosing the gain K_1 for the dominant closed-loop poles to be as close as possible to ψ.
2. Include a cascade derivative compensator of the form

$$K_2(s - z_c), \tag{5}$$

   where, initially, $K_2 = 1$ and z_c is a real zero that satisfies Eq. 4. For convenience, we repeat the two key formulas:

$$\theta_c = \pi - \angle G(\psi) \quad \text{and}$$
$$z_c = \text{Re}(\psi) - \text{Im}(\psi)/\tan\theta_c \quad (\theta_c \in [-\pi, \pi]).$$

3. Use a new root locus to tune the gain K_2 such that a closed-loop pole is at ψ.
4. Construct the closed-loop transfer function with the controller

$$K_1 K_2(s - z_c). \tag{6}$$

5. Simulate the time response to see if it meets specifications. Tune.

A design example

Let a system have plant transfer function

$$\frac{1}{(s+2)(s+6)(s+11)}. \tag{7}$$

Design a PD controller such that the closed-loop settling time is about 0.8 seconds and the overshoot is about 15%.

Determining ψ

We use Matlab for the design.[6] First, we must determine what the specified transient response criteria imply for the locations of our closed-loop poles. Let one of these desired pole locations be called ψ. The transient response performance criteria are as follows.

6. See ricopic.one/control/source/pd_controller_design_example.m for the source.

```
Ts = .8; % sec ... spec settling time
OS = 15; % percent ... spec overshoot
```

The second-order approximation from Chapter trans tells us that the settling time specification implies a specific $\text{Re}(\psi)$ and the overshoot a specific angle $\angle\psi$. The real part is found from the expressions

$$T_s = \frac{4}{\zeta\omega_n} \quad \text{and} \quad \text{Re}(\psi) = -\zeta\omega_n \Rightarrow \tag{8}$$

$$\text{Re}(\psi) = -\frac{4}{T_s}. \tag{9}$$

The angle is found via the equations

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}, \tag{10}$$

$$\tan(\angle\psi) = \frac{\sqrt{1-\zeta^2}}{\zeta}, \quad \text{and} \quad \tan(\angle\psi) = -\text{Im}(\psi)/\text{Re}(\psi). \tag{11}$$

A remarkably simple expression results:

$$\text{Im}(\psi) = -\text{Re}(\psi)\frac{\sqrt{1-\zeta^2}}{\zeta} \tag{12a}$$

$$\text{Im}(\psi) = -\text{Re}(\psi)\frac{\pi}{\ln(100/\%OS)}. \tag{12b}$$

So, in the final analysis, the desired pole location ψ (assuming the second-order approximation is valid) is given by the expression

$$\psi = -\frac{4}{T_s}\left(1 - j\frac{\pi}{\ln(100/\%OS)}\right). \tag{13}$$

This formula holds beyond the scope of this problem. We define it as an anonymous function.

```
psi_fun = @(Ts,pOS) -4/Ts*(1-1j*pi/log(100/pOS));
psi = psi_fun(Ts,OS);
disp(sprintf('psi = %0.3g + j %0.3g',real(psi),imag(psi)))
```

```
psi = -5 + j 8.28
```

P control

We design a proportional controller that gets us as close as possible to ψ. The root locus is shown in Figure PD.3.

```
G = zpk([],[-2,-6,-11],1);
figure
rlocus(G)
```

Although we cannot get close to ψ on the root locus, we can at least meet our %OS specification by choosing a gain of about

$$K_1 = 240. \tag{14}$$

Let's construct the compensator and corresponding closed-loop transfer function G_p for gain control.

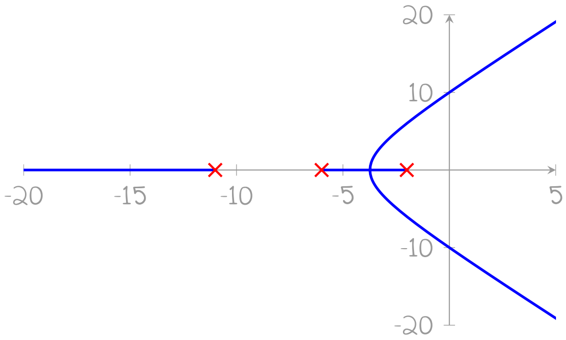

Figure PD.3: root locus without compensation.

```
K1 = 240;
G_P = feedback(K1*G,1);
```

Derivative compensation

Now, we use cascade derivative compensation with compensator

$$K_2(s - z_c). \tag{15}$$

For now, we set $K_2 = 1$. From Equation 4, we compute the compensator zero

$$z_c = \mathrm{Re}(\psi) - |\mathrm{Im}(\psi)| / \tan\theta_c \quad \text{and} \quad \theta_c = \pi - \angle G(\psi).$$

```
theta_c = pi - angle(evalfr(G,psi));
z_c = real(psi) - abs(imag(psi))/tan(theta_c);
disp(sprintf('theta_c = %0.3g deg',rad2deg(theta_c)))
disp(sprintf('z_c = %0.3g',z_c))
```

```
theta_c = 67.1 deg
z_c = -8.5
```

Let's construct the compensator sans tuned gain K_2 and tune it up using another root locus.

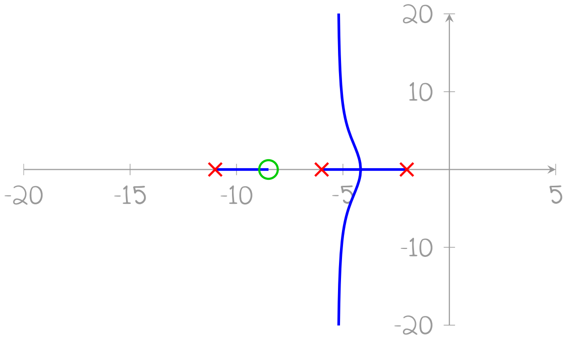

Figure PD.4: root locus with compensation.

```
C_sans = zpk(z_c,[],1);
figure
rlocus(K1*C_sans*G)
```

The resulting root locus of Figure PD.4 intersects ψ! (I mean, we knew it would, but we had our doubts.) The corresponding gain is, from Equation 2 (or we could use the data cursor),

$$K_2 = \frac{1}{|(\psi - z_c)G(\psi)|}. \tag{16}$$

Let's compute it, the controller C_{PD}, and the closed-loop transfer function G_{PD}.

```
K2 = 1/abs(evalfr(K1*C_sans*G,psi));
C = K1*K2*C_sans;
G_PD = feedback(C*G,1);
```

Simulate

Our placement of the ψ depended on the second-order approximation's accuracy, which in this case is questionable, due to the proximity of a third closed-loop pole. In any case, we simulate the step response to test the efficacy

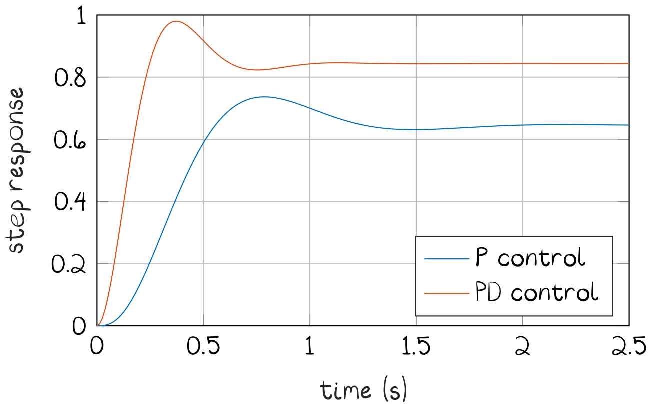

Figure PD.5: step responses for proportional and proportional-derivative controllers.

of the PD controller design and to compare it with the P controller.

```
t_a = linspace(0,2.5,200); % s ... sim time
y_P = step(G_P,t_a); % P controlled step response
y_PD = step(G_PD,t_a); % PD controlled step response
```

```
figure
plot(t_a,y_P);
hold on;
plot(t_a,y_PD);
xlabel('time (s)');
ylabel('step response');
grid on
legend('P control','PD control','location','southeast');
```

The responses, shown in Figure PLag.3, suggest the PD controller is at least close to meeting the transient specifications. It is a happy accident that the steady-state error also improved; derivative compensation does not always do this. Let's use stepinfo to compute more accurate transient response characteristics of the PD-controlled system.

```
si_PD = stepinfo(y_PD,t_a);
disp(sprintf('settling time: %0.3g',si_PD.SettlingTime))
disp(sprintf('percent overshoot: %0.3g',si_PD.Overshoot))
```

```
settling time: 0.82
percent overshoot: 16.2
```

This is quite close to the specification. If desired, the gain K_2 and the zero location z_c could be tuned, iteratively.