freq.nyquist Nyquist criterion

Introduction

Consider a feedback control system. Let $G(s)$ be the forward-loop transfer function and let $H(s)$ be the feedback transfer function. The **Nyquist plot** is a parametric plot of the frequency response function $G(j\omega)H(j\omega)$ of the open-loop transfer function $G(s)H(s)$.

Nyquist plot

The **Nyquist criterion** allows us to gain insight about closed-loop stability from the open-loop frequency response (Nyquist and Bode plots) and open-loop pole location. Additionally, insight into transient response and steady-state error response characteristics can be determined from Nyquist plots. In this sense, the Nyquist plot is analogous to the root-locus plot.

Nyquist criterion

A description of the Nyquist criterion

A rigorous derivation of the Nyquist criterion is beyond the scope of this work. However, a motivating description is included. Before we begin, please review complex functions, as described in Appendix A.01.

The full Nyquist plot is the mapping of a contour Γ_N that contains the right-half plane and is defined as beginning at the origin, moving vertically along the $j\omega$-axis "to infinity," encircling the right-half-plane with a semicircle "to negative infinity," and returning vertically to the origin, as shown in Fig. nyquist.1.

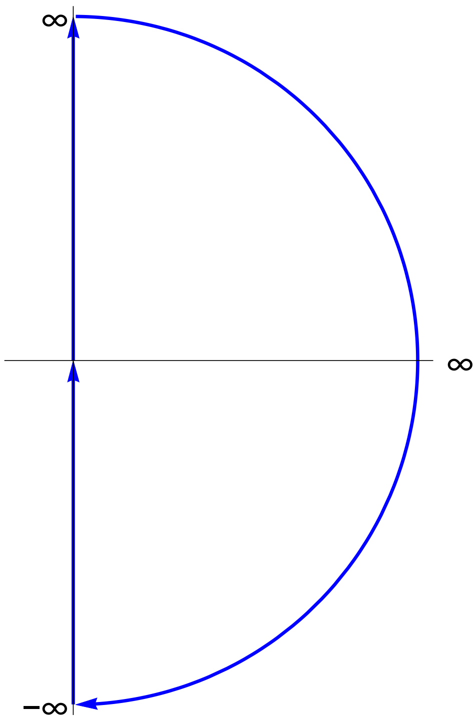

Figure nyquist.1: contour Γ_N to be mapped by transfer function.

The Nyquist plot is now defined, but what remains is describing the Nyquist criterion and why it works. There are several important insights required to understand it.

encirclements of the origin give us a clue It turns out that whenever we map Γ_N with a transfer function F, we find out a relationship between the number of poles P inside Γ_N, the number of zeros Z inside Γ_N, and the number of counterclockwise encirclements N of the origin by the contour $F(\Gamma_N)$. The primary insight is that when a pole or zero is encircled by Γ_N, it contributes an entire $\pm 2\pi$ in phase around the contour, whereas when a pole or zero is not encircled by Γ_N, its net contribution to phase is zero. This yields the following relationship:

$$N = P - Z. \tag{1}$$

the open-loop transfer function mapping is close to what we want We know the poles and zeros of the open-loop transfer function $G(s)H(s)$. We want to know information about the closed-loop pole locations. The closed-loop transfer function (without compensation) is

$$T(s) = \frac{G(s)}{1 + G(s)H(s)}. \tag{2}$$

Let's rewrite $G(s)$ and $H(s)$ in terms of numerators and denominators, as follows:

$$G(s) = \frac{G_n(s)}{G_d(s)} \quad \text{and} \quad H(s) = \frac{H_n(s)}{H_d(s)}. \tag{3}$$

Let's see what our closed-loop transfer function looks like now:

$$T(s) = \frac{G_n(s)H_d(s)}{G_d(s)H_d(s) + G_n(s)H_n(s)}. \tag{4}$$

Finally, let's consider the denominator Eq. 2 for a moment:

$$1 + G(s)H(s) = \frac{G_d(s)H_d(s) + G_n(s)H_n(s)}{G_d(s)H_d(s)}, \tag{5}$$

which, combined with Eq. 3 and Eq. 4, allows us to see two important obervations:

1. the poles of $1 + G(s)H(s)$ equal the poles of $G(s)H(s)$ and
2. the zeros of $1 + G(s)H(s)$ equal the poles of $T(s)$.

We are so close! We know the poles of $G(s)H(s)$, therefore we know the poles of $1 + G(s)H(s)$. We want to know the poles of $T(s)$, which are related to the zeros of $1 + G(s)H(s)$, which we don't have, but we have something related: the open loop transfer function mapping $G(\Gamma_N)H(\Gamma_N)$.

a sidestep for all the money What if we just map with the open-loop transfer function $G(\Gamma_N)H(\Gamma_N)$? That gives us almost exactly the same image as $1 + G(\Gamma_N)H(\Gamma_N)$, but shifted one unit to the left. This means that if we **plot** $G(\Gamma_N)H(\Gamma_N)$ and **interpret** it as $1 + G(\Gamma_N)H(\Gamma_N)$, we can determine stability of the closed loop transfer function! Let's redefine our N-P-Z relationships for the mapping $G(\Gamma_N)H(\Gamma_N)$.

1. Let N be the number of counterclockwise encirclements of -1.
2. Let P be the number of open-loop poles in the right-half plane.
3. Let Z be the number of closed-loop poles in the right-half plane.

If we have a plot of $G(\Gamma_N)H(\Gamma_N)$, we have the first two and the third is given by the **Nyquist criterion**:

$$Z = P - N. \tag{6}$$

We will use this to determine stability in Lec. freq.nystab. However, even now, we know that the existence of right-half-plane closed-loop poles implies closed-loop instability, so we can already identify that much. Before exploring stability further, we will learn to sketch the Nyquist plot. Not because we don't have MATLAB, but rather to gain intuition.

Sketching Nyquist plots

We now begin sketching Nyquist plots. Remember that we are first-of-all interested in the number of counterclockwise encirclements of -1, which will help us determine stability via the Nyquist criterion. We proceed by example.

Example freq.nyquist-1 re: a stable open-loop system

Let an open-loop transfer function be defined by

$$G(s)H(s) = \frac{30}{(s+4)(s+7)}.$$

Sketch its Nyquist plot and apply the Nyquist criterion to determine the number of closed-loop poles in the right-half plane.

Let's sketch the contour Γ_N in Fig. nyquist.2.

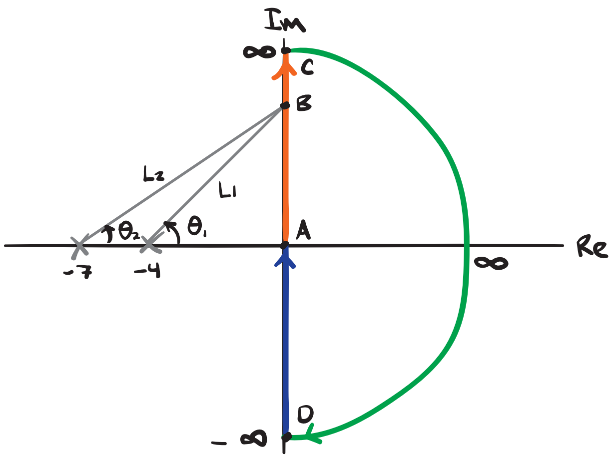

Figure nyquist.2:

So the magnitude and phase of the mapped contour are

We begin at point A and map the orange contour, which is the positive $j\omega$-axis. At A, $\theta_1 = \theta_2 = 0$, so $\angle G(A)H(A) = 0$, and $L_1 = 4$ and $L_2 = 7$, so $|G(A)H(A)| = 30/28 \approx 1.07$. In the Fig. nyquist.3, we sketch $G(\Gamma_N)H(\Gamma_N)$ with point $A' = G(A)H(A)$. As we move to B on the orange contour, the angle becomes increasingly negative and the magnitude decreases. Finally, at C, the angle approaches -180 deg and the magnitude approaches 0. Note that in the sketch we don't go quite to zero because we want to leave space to represent what occurs at zero.

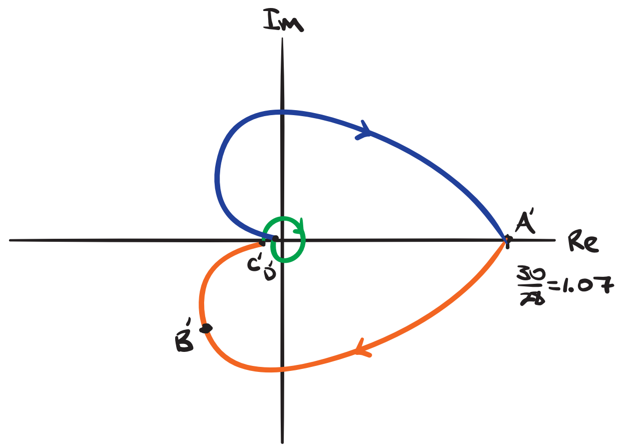

Figure nyquist.3:

What occurs at zero is that the green contour "at infinity" is mapped. The angle changes from +180 deg to -180 deg and the magnitude stays at 0. We sketch this by showing a 360 deg rotation back to +180 deg = -180 deg at C′. This doesn't always happen. Sometimes the angle with which the origin is approached is different than the angle with which it leaves. In this case, the blue contour exits at 180 deg with increasing amplitude, only to "mirror" the orange contour's return to A′. This **does** always occur: The Nyquist plot is always symmetric about the real axis and the $j\omega$-axis image is essentially a mirroring of the $-j\omega$-axis image.

Examining the Nyquist plot sketch, there are no counterclockwise encirclements of -1, i.e. N = 0. The open-loop transfer function has no poles in the right-half-plane, i.e. P = 0. Therefore, from the Nyquist criterion, So there are no closed-loop in the right-half-plane and the closed-loop system is **stable**.

What if there's an open-loop pole on the contour Γ_N? The magnitude of the contour $G(\Gamma_N)H(\Gamma_N)$ becomes infinite, but we cannot determine at which phase it does so. Therefore, in these cases we take an infinitesimal detour around the pole so that we can keep track of the phase. The magnitude still approaches infinity, but the phase information is retained. Let's consider another example that illustrates this.

Example freq.nyquist-2 re: a system with open-loop poles on the Nyquist contour

Let an open-loop transfer function be defined by

$$G(s)H(s) = \frac{10(s+1)}{s^2+1}.$$

Sketch its Nyquist plot and apply the Nyquist criterion to determine the number of closed-loop poles in the right-half plane.

Let's sketch the contour Γ_N in Fig. nyquist.4.

So the magnitude and phase of the mapped contour are

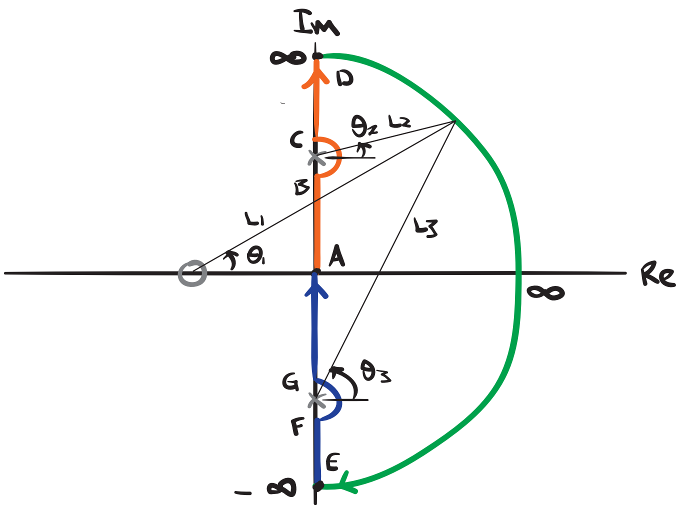

Figure nyquist.4:

We begin at point A and map the orange contour, which is the positive $j\omega$-axis. At A, $\theta_1 = 0$ and $\theta_2 = -\theta_3$, so $\angle G(A)H(A) = 0$, and $L_1 = L_2 = L_3 = 1$, so $|G(A)H(A)| = 10$. In Fig. nyquist.5, we sketch $G(\Gamma_N)H(\Gamma_N)$ with point $A' = G(A)H(A)$. As we move to B on the orange contour, $\theta_1 \to +45$ deg and still $\theta_2 = -\theta_3$, so $\angle G(A)H(A) \to +45$ deg. But the magnitude approaches infinity because $L_2 \to 0$. The infinitesimal detour from B to C doesn't change the magnitude, but it does change the phase by -180 deg. Finally, from C to D the only angle that changes is θ_1 by +45 deg, which yields $\angle G(A)H(A) \to -90$ deg as the magnitude approaches zero due to the denominator of the magnitude approaching infinity faster than the numerator. What occurs at zero is that the green contour "at infinity" is mapped. The angle changes from -90 deg to +90 deg and the magnitude stays at 0. We sketch this by showing a 180 deg rotation back to +90 deg at C'. The blue contour exits at +90 deg with increasing amplitude, only to "mirror" the orange contour's return to A'.

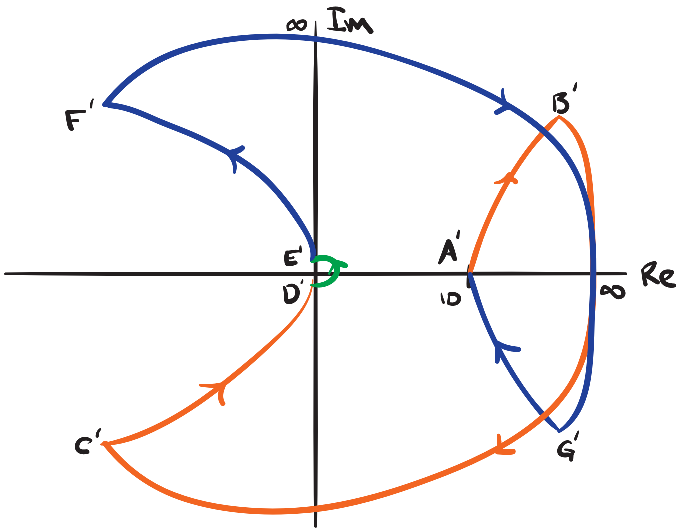

Figure nyquist.5:

Examining the Nyquist plot sketch, there are no counterclockwise encirclements of -1, i.e. N = 0. The open-loop transfer function has no poles in the right-half-plane, i.e. P = 0. Therefore, from the Nyquist criterion, So there are no closed-loop in the right-half-plane and the closed-loop system is **stable**.