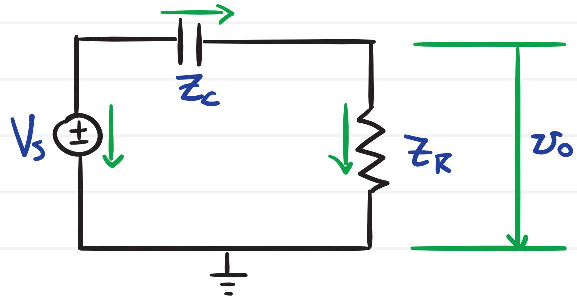
Complex circuit analysis of a differentiator (example)

Given the differentiator circuit shown, with $V_s(t) = A\cos(\omega t)$, what is v_o in the steady-state?



1. Sign convention.

2. Since there is a sinusoidal input and we are concerned about steady-state behavior, we can use impedance analysis. The elemental equation of each passive element is given by the generalized Ohm's law:

— Resistor: $v_R = i_R Z_R = i_R R$

— Capacitor: $v_C = i_C Z_C = i_C \frac{1}{\omega C} e^{-j\frac{\pi}{2}}$.

3. KCL: $i_C = i_R = i$

4. KVL: $V_s = v_C + v_R$

5. Complex algebra. We want $v_o = v_R$. Therefore,

$$v_o = V_s - v_C = A - i\frac{1}{\omega C}e^{-j\frac{\pi}{2}} = A - \frac{v_o}{R}\frac{1}{\omega C}e^{-j\frac{\pi}{2}}$$

$$\Rightarrow \quad v_o = A\Big/\left(1 + \frac{1}{RC\omega}e^{-j\frac{\pi}{2}}\right) = A\Big/\left(1 + \frac{1}{RC\omega}\left(\cos\frac{-\pi}{2} + j\sin\frac{-\pi}{2}\right)\right)$$

$$= A\Big/\left(1 - j\frac{1}{RC\omega}\right) = A\Big/\left(\sqrt{1^2 + \left(\frac{1}{RC\omega}\right)^2}\, e^{j\phi}\right) \quad \Bigg| \quad \phi \equiv \arctan\left(\frac{-\frac{1}{RC\omega}}{1}\right)$$

$$= \frac{ARC\omega}{\sqrt{(RC\omega)^2 + 1}} e^{-j\phi} = \frac{A\omega}{\sqrt{\omega^2 + 1/(RC)^2}} \cos(\omega t - \phi)$$.