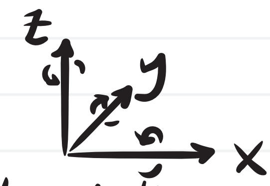
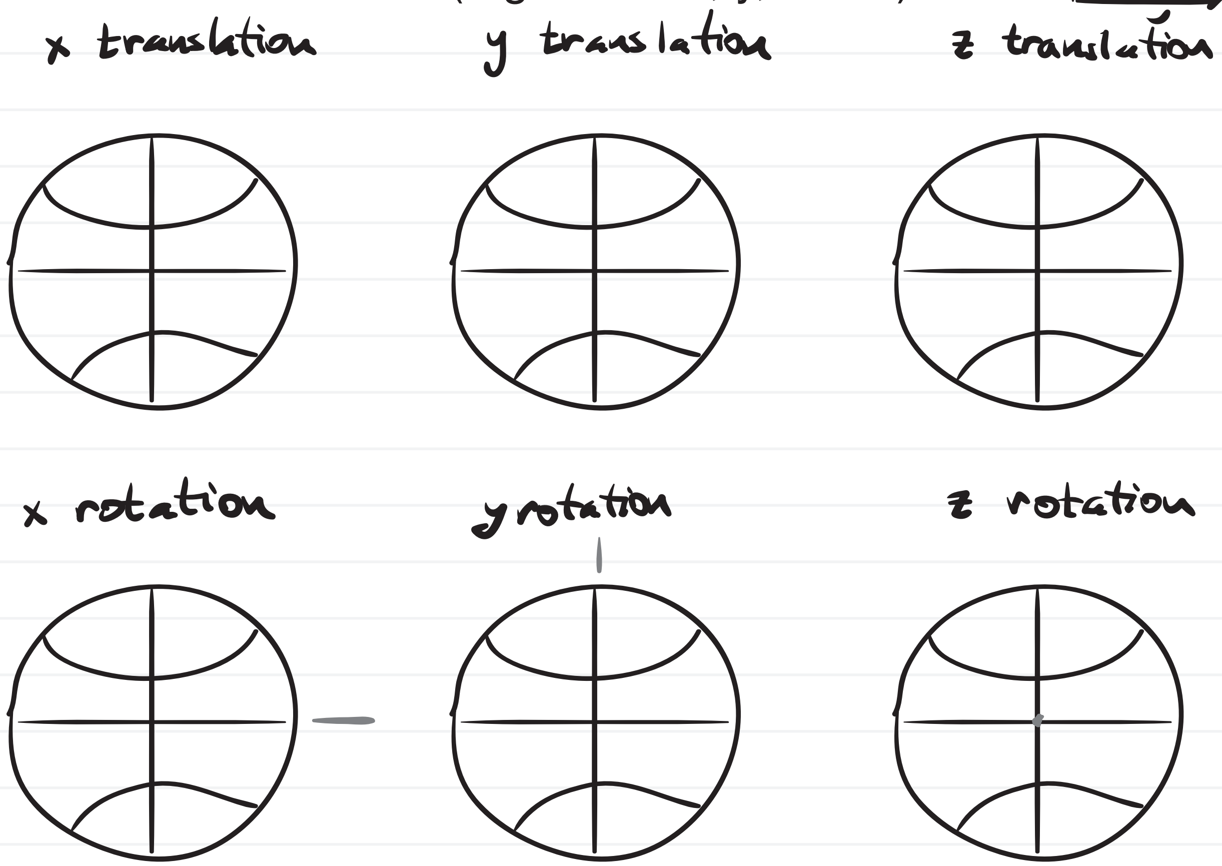
Moments, couples, and basketballs

Although we are focusing on static bodies in this class, we should keep in mind how a rigid body can move in space. We say that an object has six degrees of freedom:
translation in three directions (e.g. x, y, and z) and
rotation in three directions (e.g. about x, y, and z).





We know that, in order to get the rigid body to move, we must apply forces. Each point on the body will respond to these forces according to Newton's second law:

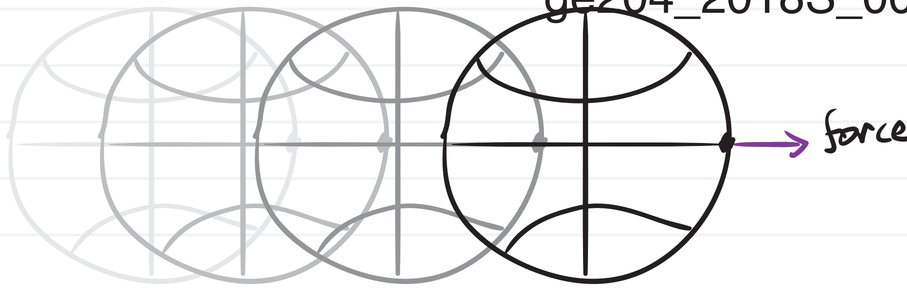
Since the body is rigid, each point is constrained relative to the others. For instance, on a Spalding basketball, the letters do not move relative to each other.

When we assume a body is rigid, we assume internal forces will maintain those constraints. For the basketball, we assume the rubber and leather and air pressure inside maintain the same relative positions of the letters “p” and “n”. That is, they transmit forces -- for the solid materials, tension and compression -- such that “p” and “n” stay in the same relative positions.

I say “relative” positions because, of course, when I spin the ball on my finger, both the “p” and the “n” are not “fixed” points. By the way, what do we mean by a “fixed” point? In an absolute sense, there is no such thing! This is one of the fundamental concepts of mechanics, that every motion is relative motion! However, Newtonian mechanics requires a so-called inertial coordinate system, which is one that is not accelerating. Sometimes we pretend that such a system is “fixed,” by which we mean it is taken to be stuck to some point in space or some specific body that we will take as being “fixed” (like the ground).

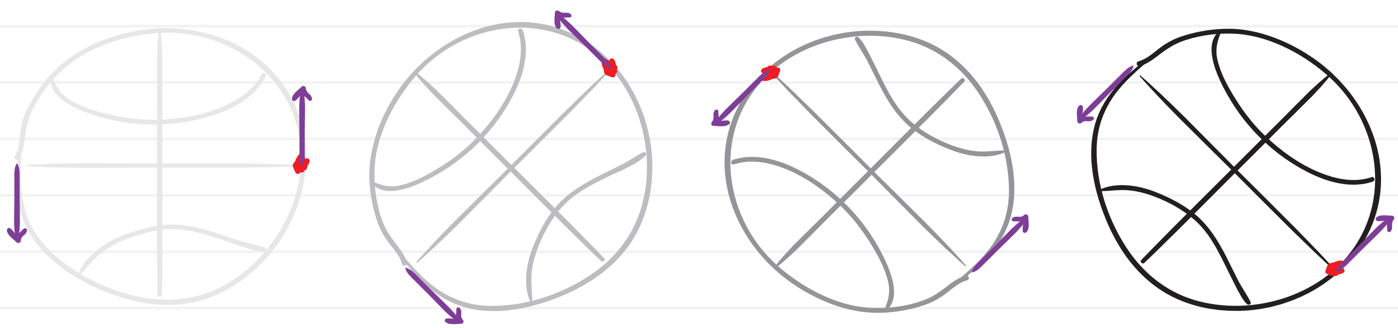
From a “fixed” reference, let’s say the floor, in our case, we observe when I spin the ball that, while the *relative* positions of the “p” and the “n” are fixed, the positions from our fixed reference surely is not!

Let’s imagine our ball is stationary (relative to some reference point) and floating in space sufficiently far away from any planets such that no gravity acts on the ball. Newton tells us the ball will remain stationary until a force accelerates it. Let’s imagine applying such a force at a single point, as shown.



All of the points moved together, rightward. All the internal forces orchestrated this movement of all the points, in unison. This is pure translation: the body did not rotate!

Now we apply two forces “tangential” to the surface of the ball.



It rotates, but does not translate! It does not translate because the external forces sum to zero. But how, then, does it rotate? Each point in the ball must follow Newton’s second law, so it must respond to the forces applied. Each point “pushes” and “pulls” those around it, accordingly. In this case, the resulting motion of the rigid body is a pure rotational motion about an axis perpendicular to the board, through the center of the ball. We call this type of loading that causes rigid body rotation a torque.

If the forces are equal in magnitude and opposite in direction (as in our second example), we call these two forces a couple. If they are-applied to the same point on the rigid body, these forces completely cancel and create no torque. In fact, if the are applied colinearly, they would cancel and create no torque. But if they are applied in any other orientation, they do create a torque. Another word used for torque is moment.

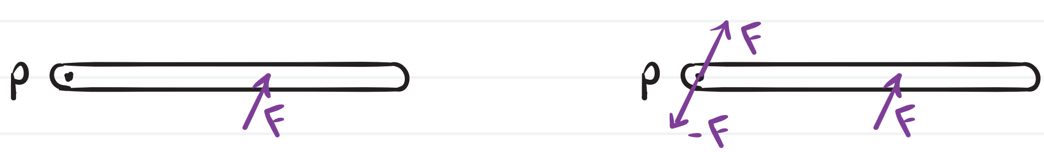
This moment can be considered a rotational analog for force. It is beyond the scope of this course to consider the general three-dimensional equations of motion, but if we assume all motion to be in a plane, then the rotational analog of Newton's second law is written

In the static equilibrium case, the general three-dimensional force and moment equations can be easily written

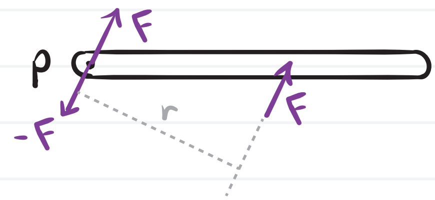
It is only barely an exaggeration to say that these two equations are pretty much this entire course!

However, we don't really yet understand how to use them. Probably the most glaring lack in our understanding is this: how can we quantify what the moment of a given force is? It is to this question that we now turn.

A force creates a moment that is generally different for each point P in the rigid body. Remember how our couple couldn't be colinear to create a moment? It is this distance between them that is the special sauce. Consider the following link loaded with a force F.



Trickery! We have added a net zero force at P, which we can always do (just like how we can always multiply by one!), and now we have a couple! It turns out that the couple creates a moment that is proportional to the perpendicular distance between the forces.



In fact, the expression for the magnitude of the moment about point P is simply:

But a moment also has a direction: perpendicular to the plane in which the couple resides. So, if the forces are in the x-y plane,

But what if they are in some other plane? There is a mathematical operation that simplifies this sort of analysis a great deal: the cross product “x”. Let **r** be the vector from point P to the location the force **F** is applied. Then the moment of **F** about P is

If we wanted to know how much moment of **F** is about an axis with unit vector **u**, then we can use another mathematical operator on the result: the dot product “•”:

This is sometimes called the triple product.

We will now take some time to do examples, but before we do, let’s consider that if multiple forces are applied at the same point, we can say they sum to **F** and therefore the equations we’ve developed still apply!