General uncertainty analysis

Let R be a result (r.v. too) that depends on J measurands X_i. The absolute sensitivity coefficient θ_i for each measurand is defined as

$$\theta_i \equiv \frac{\partial R}{\partial X_i} .$$

Example Given a current measurement through a resistor with resistance Ω, what is the absolute sensitivity coefficient for the resulting voltage?

The result is $v = \Omega i$. Therefore, $\theta_1 = \frac{\partial v}{\partial i} = \Omega$.

In the general situation of both systematic and random errors, the combined estimated uncertainty in the result u_R is defined by

$$u_R^2 \equiv \sum_{i=1}^{J} \theta_i^2 S_{Bi}^2 \quad \text{(where } S_{Bi} \text{ are the (sample) stds of the systematic errors)}$$

$$+ 2 \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \theta_i \theta_j S_{Bi,Bj} \quad \text{(where } S_{Bi,Bj} \text{ are the covs of the systematic errors)}$$

$$+ \sum_{i=1}^{J} \theta_i^2 S_{Pi}^2 \quad \text{(where } S_{Pi} \text{ are the stds of the random errors)}$$

$$+ 2 \sum_{i=1}^{J-1} \sum_{j=i+1}^{J} \theta_i \theta_j S_{Pi,Pj} \quad \text{(where } S_{Pi,Pj} \text{ are the covs of the random errors).}$$

We usually want the overall uncertainty, defined by $U_R^2 \equiv t_{\nu_R,C}^2 \cdot u_R^2$, where ν_R is the number of effective degrees of freedom. It is

$$\nu_R = \frac{\left(\sum_{i=1}^{J} \left(\theta_i^2 S_{Bi}^2 + \theta_i^2 S_{Pi}^2 \right) \right)^2}{\sum_{i=1}^{J} \left(\frac{\theta_i^4 S_{Pi}^4}{\nu_{Pi}} + \sum_{k=1}^{M_B} \frac{\theta_i^4 (S_{Bi})_k^4}{\nu_{(S_{Bi})_k}} \right)}$$

J measurands

M_B systematic errors for each measurand

$\nu_{Pi} = N_i - 1$ # of measurements for ith measurand

$\nu_{(S_{Bi})_k} \cong \frac{1}{2} \left(\frac{\Delta (S_{Bi})_k}{(S_{Bi})_k} \right)^{-\frac{1}{2}}$

Example Given a sample of each of the measurands voltage v across a given resistor and current i through it, estimate the resistance Ω of the resistor. Assume the measurements are uncorrelated. The multimeters used each have the following specs.

Voltage accuracy: 1 %.
Current accuracy: 2 mA.

The data is as follows.

measurement	voltage (V)	current (A)
1	6.13	0.541
2	6.21	0.527
3	6.18	0.533
4	6.19	0.537
5	6.13	0.529
6	6.22	0.543
7	6.23	0.532
8	6.18	0.535
9	6.20	0.540
10	6.12	0.535

1. Sample means: $\bar{v} = 6.179\ \text{V}$ $\qquad \bar{i} = 0.535\ \text{A}$

2. Sample stds: $S_v = 0.0396\ \text{V}$ $\qquad S_i = 0.00518$

3. SDOMs: $S_{\bar{v}} = S_v/\sqrt{N} = 0.0125\ \text{V}$ $\qquad S_{\bar{i}} = S_i/\sqrt{N} = 0.0164$

4. Random uncertainty: identify that $S_{P_{\bar{v}}} = S_{\bar{v}}$ and $S_{P_{\bar{i}}} = S_{\bar{i}}$

5. Accuracies: $S_{B_v} = (1\%)\bar{v} = 61.79\ \text{mV}$ $\qquad S_{B_i} = 2\ \text{mA}$

6. Sensitivity: $v = \Omega i \Rightarrow \Omega = \frac{v}{i} \Rightarrow \theta_v = \frac{\partial \Omega}{\partial v} = \frac{1}{i}$ & $\theta_i = \frac{\partial \Omega}{\partial i} = -\bar{v}\,\bar{i}^{-2}$

7. Combined estimated uncertainty:

$$u_\Omega = \left(\theta_{\bar{v}}^2 S_{B\bar{v}}^2 + \theta_i^2 S_{Bi}^2 + \theta_{\bar{v}}^2 S_{P\bar{v}}^2 + \theta_i^2 S_{Pi}^2 \right)^{1/2}$$

$$= 0.130 \text{ Ohms}.$$

The mean value for Ω is $\bar{\Omega} = \bar{v}/\bar{i} = 11.5$ Ohms.
So our uncertainty is relatively small.
Let's assign a confidence interval.

8. The overall uncertainty with 95% confidence is

$$U_\Omega = t_{\nu_\Omega, 95} \cdot u_\Omega .$$

Let's compute ν_Ω.

$$\nu_\Omega = \frac{\left(\sum_{i=1}^{J} \left(\theta_i^2 S_{Bi}^2 + \theta_i^2 S_{Pi}^2 \right) \right)^2}{\sum_{i=1}^{J} \left(\frac{\theta_i^4 S_{Pi}^4}{\nu_{Pi}} + \sum_{k=1}^{M_B} \frac{\theta_i^4 (S_{Bi})_k^4}{\nu_{(S_{Bi})_k}} \right)}$$ where we have

$J = 2$ measurands,
$M_B = 1$ systematic errors for each measurand,
$\nu_{P\bar{v}} = N_{\bar{v}} - 1 = 10 - 1 = 9$, $\nu_{Pi} = N_i - 1 = 10 - 1 = 9$
the assumption of 100% certainty in given instrumental accuracies,
(and therefore)

$$\nu_\Omega = \frac{\left(\theta_{\bar{v}}^2 S_{P\bar{v}}^2 + \theta_i^2 S_{Pi}^2 \right)^2}{\frac{\theta_{\bar{v}}^4 S_{P\bar{v}}^4}{\nu_{P\bar{v}}} + \frac{\theta_i^4 S_{Pi}^4}{\nu_{Pi}}}$$

$$= 15.6 .$$

This corresponds to approximately $t_{16,95} = 2.120$. Therefore,

$U_\Omega = (2.120)(0.130) = 0.276$ Ohms. So $m_\Omega = 11.5 \pm 0.276$ Ohms with a confidence of 95%.