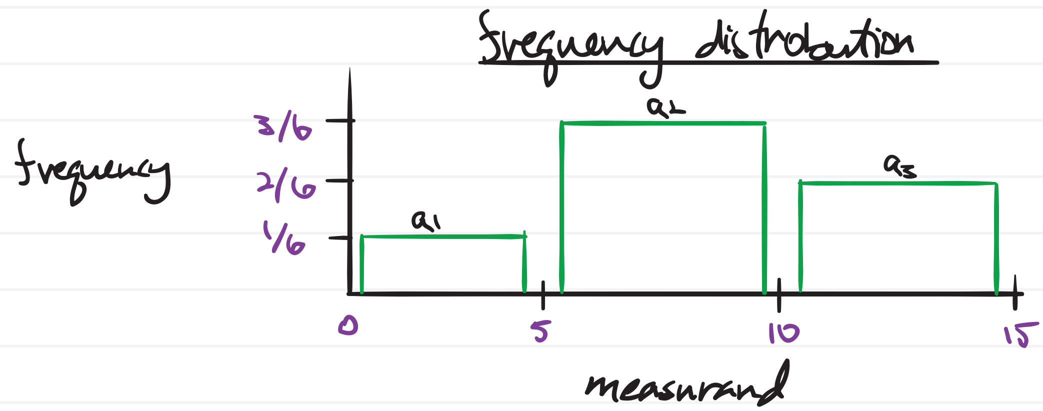
Probability density functions + probability mass functions

Say an experiment is performed during which the measurand is recorded through a range of values. We can plot the relative frequency of the measurand landing in different "bins" — ranges of values. This is called a frequency distrob. For instance,



The frequencies a_i must sum to unity: $\sum_{i=1}^{K} a_i = 1$. The frequency density distrobution is similar to the frequency distro., but with $a_i \mapsto a_i/\Delta x$, where Δx is the interval width.

If we have smaller and smaller intervals in our frequency density distrobution, we can derive a probability density function:

$$f(x) = \lim_{N\to\infty,\, \Delta x\to 0} \sum_{j=1}^{K} a_j/\Delta x .$$

We typically think of a probability density function as a function that can be integrated over to find the probability of the random variable being in an interval:

Of course, $\Pr[-\infty < X < \infty] = \int_{-\infty}^{\infty} f(x)\,dx = 1$.

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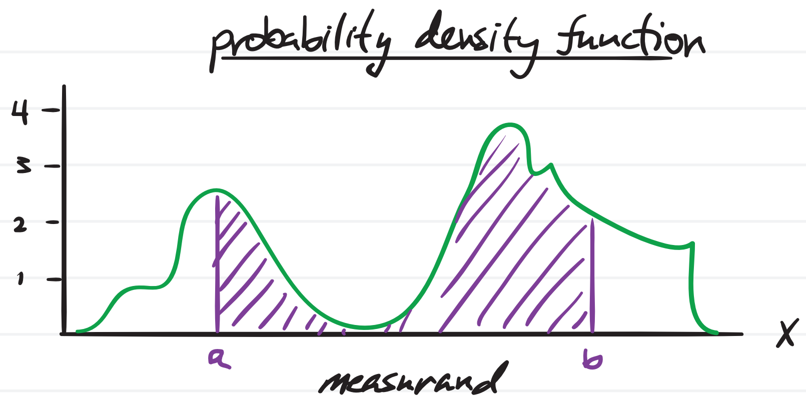


We will now look at some PMFs and PDFs.

Binomial PMF

Consider a binary sequence of length n with each element a random 0 or 1 generated independently, like

$$1 \quad 0 \quad 1 \quad 1 \quad 0 \quad \cdots \quad 0 \quad 1 .$$

This sequence has the probability of occurring:

$$\overbrace{p}^{\text{prob. of a 1}} \cdot (1-p) \cdot p \cdot p \cdot (1-p) \cdots (1-p) \cdot p$$

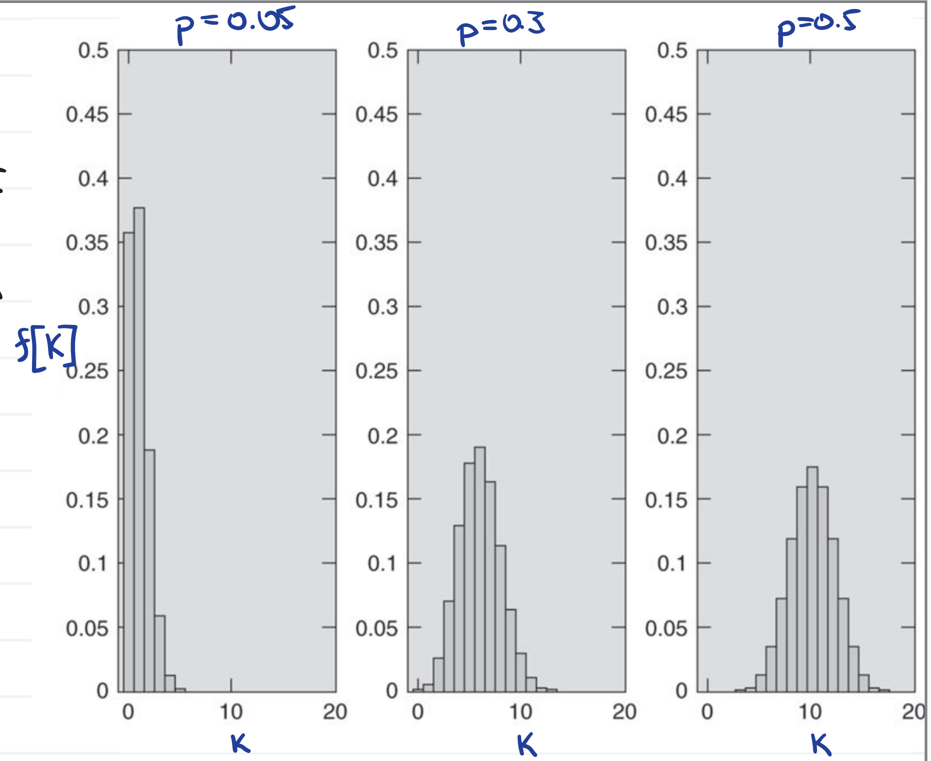
$$= p^{\overbrace{k}^{\text{number of 1s.}}} (1-p)^{(n-k)}$$

There are $\binom{n}{k}$ (n choose k $= \frac{n!}{k!(n-k)!}$) possible combinations of k 1s in n bits. Therefore, the probability of any combination of k 1s in a series is:

$$f[k] = \binom{n}{k} p^k (1-p)^{n-k},$$

which is the binomial distribution PMF.

Example Consider a field sensor that fails for a given measurement with probability p. Given n measurements, we can plot the Binomial PMF for different probabilities of failure (right).

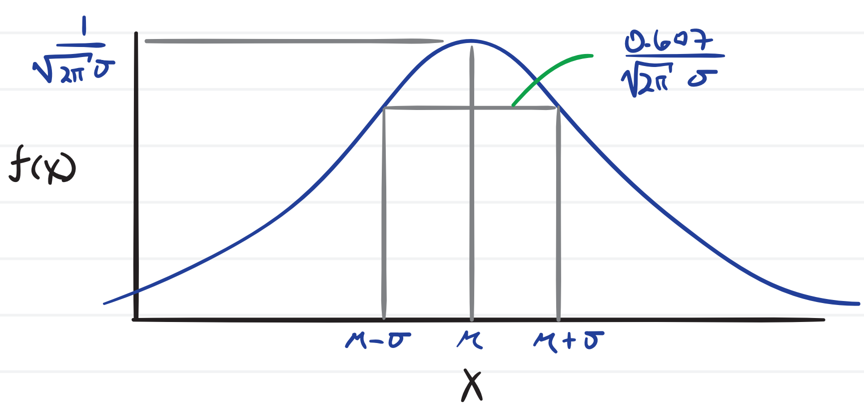


Gaussian PDF ("Normal" distribution)

The Gaussian or normal random variable has PDF:

where:
μ is the mean
σ is the standard deviation
σ^2 is the variance

This has the familiar "bell" shape:



It can be shown that $Pr[\mu-\sigma < X \leq \mu+\sigma] = 68.2\%$.

Noise is often modelled as Gaussian. Many random processes are nearly Gaussian, so it is important to understand.