Solution for exercise 06.1 trans.

25 p.

a. The ODE is given in the standard form of a first-order ODE; therefore, by inspection, the time constant $\tau = 7$.

b. From Table firsto.1, for a unit ramp function, the characteristic response is

$$y_r = t - \tau(1 - e^{-t/\tau}).$$

c. Beginning with y_r and applying superposition and the derivative property, the forced response is

$$\begin{aligned} y_{fo} &= \dot{y}_r - 5y_r \\ &= (1 - e^{-t/\tau}) - 5(t - \tau(1 - e^{-t/\tau})) \\ &= 1 + 5\tau - 5t - (1 + 5\tau)e^{-t/\tau}. \end{aligned}$$

d. The free response y_{fr} to initial condition $y(0) = 8$ is given by Eq. 2 to be

$$y_{fr}(t) = y(0)\, e^{-t/\tau}.$$

e. The total response y_t when both the input u and initial condition $y(0)$ are applied simultaneously is, by superposition, merely the sum

$$\begin{aligned} y_t &= y_{fr} + y_{fo} \\ &= y(0)\, e^{-t/\tau} + 1 + 5\tau - 5t - (1 + 5\tau)e^{-t/\tau} \\ &= 1 + 5\tau - 5t - (1 + 5\tau - y(0))e^{-t/\tau} \\ &= 1 + 5\tau - 5t + (7 - 5\tau)e^{-t/\tau} \\ &= 36 - 5t - 28e^{-t/7}. \end{aligned}$$

Solution for exercise 06.2 trans.

a. The ODE is already in standard form; therefore, identify

$$\begin{aligned}\omega_n &= \sqrt{25} \\ &= 5. \\ \zeta &= \frac{5}{2\omega_n} \\ &= 1/2.\end{aligned}$$

b. From Table secondo.1 for damping $\zeta \in (0, 1)$ and forcing $f(t) = u_s(t)$,

$$y_{ch}(t) = \frac{1}{\omega_n^2}\left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t + \psi)\right)$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and $\psi = -\arctan(\zeta/\sqrt{1-\zeta^2})$.

c. From superposition, the forced response is

$$y_{fo}(t) = 2\dot{y}_{ch} + 3y_{ch},$$

where

$$\begin{aligned}\dot{y}_{ch} &= \frac{e^{-\omega_n\zeta t}}{\omega_n^2\sqrt{1-\zeta^2}}\left(\omega_d \sin\left(\omega_d t + \psi\right) + \omega_n\zeta\cos\left(\omega_d t + \psi\right)\right) \\ &= \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t).\end{aligned}$$

This last equality can be derived by applying a two-to-one formula from **??** or recognizing that $\dot{u}_s = \delta$ and using the characteristic response formula corresponding to δ in Table secondo.1.

Solution for exercise 06.3 trans.

20 p.

a. The ODE is not given in the standard form of a first-order ODE. Dividing both sides by 3,

$$\frac{1}{3}\dot{y} + y = \frac{2}{3}\dot{u} + \frac{1}{3}u. \tag{1}$$

Therefore, by inspection, the time constant $\tau = 1/3$.

b. From Table firsto.1, for a unit step input, the characteristic response is

$$y_s = 1 - e^{-t/\tau}.$$

c. Beginning with y_s and applying superposition and the derivative property, the forced response is

$$\begin{aligned} y_{fo} &= 3\left(\frac{2}{3}\dot{y}_s + \frac{1}{3}y_s\right) \\ &= 2\dot{y}_s + y_s \\ &= 2 \cdot 3e^{-3t} + 1 - e^{-3t} \\ &= 1 + 5e^{-3t}. \end{aligned}$$

d. The free response y_{fr} to initial condition $y(0) = -4$ is given by Eq. 2 to be

$$y_{fr}(t) = y(0)\, e^{-t/\tau}.$$

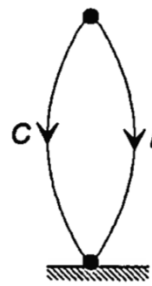
e. The total response y_t when both the input u and initial condition $y(0)$ are applied

simultaneously is, by superposition, merely the sum

$$\begin{aligned} y_t &= y_{fr} + y_{fo} \\ &= y(0)\, e^{-t/\tau} + 1 + 5e^{-3t} \\ &= -4\, e^{-3t} + 1 + 5e^{-3t} \\ &= 1 + e^{-3t}. \end{aligned}$$

Problem 9.9

The linear graph for the system (with the switch open) is:



The system has the state equation:

$$\dot{v}_C = -\frac{1}{RC} v_C$$

and a time constant $\tau = RC$ s. The decay in voltage due to leakage is

$$v_C(t) = V(0)e^{-t/RC}$$

At time $t = 100$ s, $3.68 = 10e^{-100/(R \times 5 \times 10^6)}$ or

$$R = \frac{10^8}{5 \ln(10/3.68)} = 20 \times 10^6 \Omega = 20\text{M}\Omega$$

Time (secs)

Problem 9.19

The undamped natural frequency, and damping ratio are:

$$\omega_n = \sqrt{\frac{a_0}{a_2}}$$

$$\zeta = \frac{1}{2} a_1 \sqrt{\frac{1}{a_2 a_0}}$$

The differential equation in terms of ζ and ω_n is

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = \frac{1}{a_2} u(t)$$

To determine the units of the coefficients, the nature and units of the input $u(t)$ must be known, and are not given here. For illustration, assume that the input is a force, then the units are

Coefficient	SI Units	English Units
a_0	N/m	lb/ft
a_1	N.s/m	lb.s/ft
a_2	N.s^2/m (or kg)	lb.s^2/ft (or slug)

Generalization to other input forms is simple.

Problem 9.20

Mechanical System: The characteristic equation is

$$\lambda^2 + \frac{K}{B}\lambda + \frac{K}{m} = 0$$

from which

$$\omega_n = \sqrt{\frac{K}{m}}$$

$$\zeta = \frac{1}{2B}\sqrt{mK}$$

The undamped natural frequency may be increased by either increasing the spring stiffness K or decreasing the mass m. These two choices have opposite effects on the damping ratio: increasing K will increase ζ while decreasing m will decrease ζ. The value of the viscous coefficient B will affect the value of ζ without affecting ω_n.

Electrical System:

The characteristic equation is:

$$\lambda^2 + \left(\frac{R_2}{L} + \frac{1}{CR_1}\right)\lambda + \frac{R_1 + R_2}{R_1}\frac{1}{LC} = 0$$

The undamped natural frequency and damping ratio are:

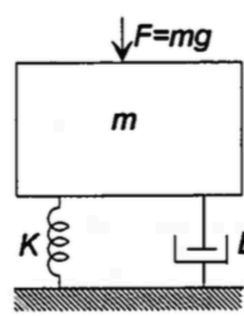
$$\omega_n = \sqrt{\frac{R_1 + R_2}{R_1}\frac{1}{LC}}$$

$$\zeta = \frac{1}{2}\left(\frac{R_2}{L} + \frac{1}{CR_1}\right)\sqrt{\frac{R_1 LC}{R_1 + R_2}}$$

In this case the values of all four system elements will affect both ω_n and ζ.

Problem 9.22

The plant model is shown below:



(a) The state equations are:

$$\begin{bmatrix} \dot{v}_m \\ \dot{F}_K \end{bmatrix} = \begin{bmatrix} -B/m & -1/m \\ K & 0 \end{bmatrix} \begin{bmatrix} v_m \\ F_K \end{bmatrix} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} F_s$$

The equivalent source is a force source $F_s = mg$.

(b) The differential equation relating mass displacement to input force is

$$\frac{d^2y}{dt^2} + \frac{B}{m}\frac{dy}{dt} + \frac{K}{m}y = g$$

The undamped natural frequency and damping ratio are:

$$\omega_n = \sqrt{\frac{K}{m}}$$
$$\zeta = \frac{B}{2}\sqrt{\frac{1}{Km}}$$

(c) The differential equation in part (b) shows that the steady-state deflection is

$$y_{ss} = \frac{mg}{K}$$

allowing K to be determined from the steady-state deflection. The value of B may be estimated by comaparing the overshoot to the standard form shown in Fig. 9.22.

Sample (a)

$$K = \frac{mg}{y_{ss}} = \frac{9.81}{0.3} = 32.7 \text{ N/m}$$

The response shows approximately 50% overshoot; from Fig. 9.22 thi indicates $\zeta \approx 0.2$. With this value

$$B = 2\zeta\sqrt{Km} = 2 \times 0.2 \times \sqrt{32.7 \times 1} = 2.29 \text{ N.s/m}$$

Sample (b)

$$K = \frac{mg}{y_{ss}} = \frac{9.81}{0.2} = 49.05 \text{ N/m}$$

The response shows that the system is over- (or at least critically-) damped. Use the sttling time to estimate ζ. The settling time is approximately 2 seconds, and $\omega_n = \sqrt{K/m} = 7.0$ rad, so that the normalized settling time is $\omega_n t = 14$. From Fig. 9.22 we estimate that $\zeta = 2.0$. Then

$$B = 2\zeta\sqrt{Km} = 2 \times 2 \times \sqrt{49.05 \times 1} = 28.01 \text{ N.s/m}$$

Problem 9.23

(a)

$$\frac{d^2y}{dt^2} + \frac{B_{eq}}{m}\frac{dy}{dt} + \frac{K_{eq}}{m}y = \frac{B_{eq}}{m}\frac{dV_s}{dt} + \frac{K_{eq}}{m}V_s$$

where $K_{eq} = 4 \times 4000 = 16000$ N/m, and $B_{eq} = 4 \times 300 = 1200$ N.m/s. With these values the equations is

$$\frac{d^2y}{dt^2} + 3.75\frac{dy}{dt} + 50y = 3.75\frac{dV_s}{dt} + 50V_s$$

The undamped natural frequency and damping ratio are:

$$\omega_n = \sqrt{\frac{K_{eq}}{m}} = \sqrt{50} = 7.07 \text{ rad/s}$$

$$\zeta = \frac{B_{eq}}{2}\sqrt{\frac{1}{mK_{eq}}} = \frac{1200}{2}\sqrt{\frac{1}{320 \times 16000}} = 0.265$$

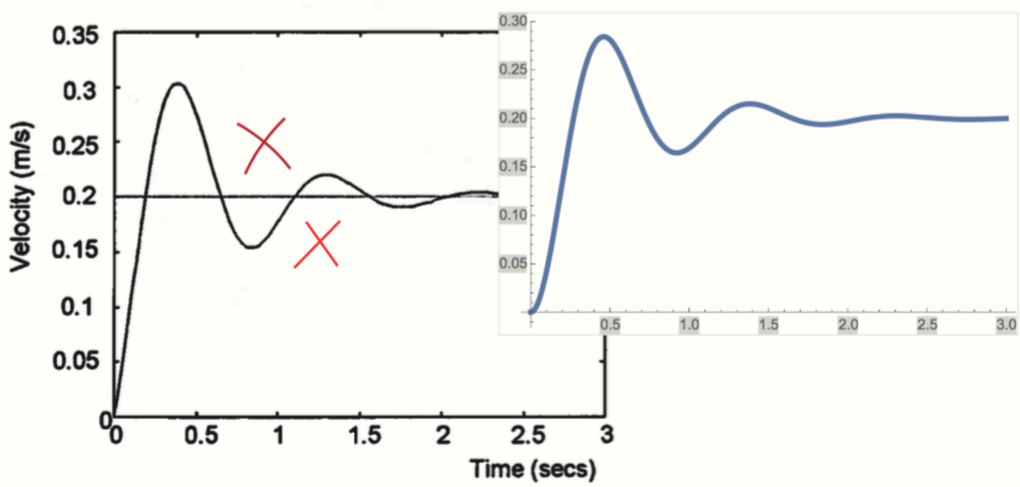
(b) The system unit step response is:

$$y_{step}(t) = 1 + 0.275e^{-1.875t}\sin(6.82t) - e^{-1.875t}\cos(6.82t)$$

so that the response to an input step of 0.2 m/s is

$$v_m(t) = 0.2 + 0.055e^{-1.875t}\sin(6.82t) - 0.2e^{-1.875t}\cos(6.82t)$$

which is shown below:



(c) To prevent oscillatory responses the system must be at least critically damped.

$$\zeta = \frac{B_{eq}}{2}\sqrt{\frac{1}{mK_{eq}}} \geq 1$$