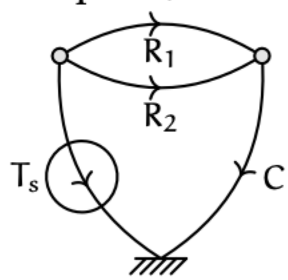
Exercise 13.1 tile

Use the linear graph below of a thermal system to **(a)** derive the transfer function $T_{R_2}(s)/T_s(s)$, where T_s is the input temperature and T_{R_2} is the temperature across the thermal resistor R_2. *Use impedance methods*. And **(b)** derive the input impedance the input T_s drives.



Solution

a. The transfer function $T_{R_2}(s)/T_s(s)$ can be found easily from an across-variable divider rule. The combined impedance of R_1 and R_2 (in parallel, i.e., $Z_{R_1} \parallel Z_{R_2}$) retains the temperature difference T_{R_2}, so

$$\frac{T_{R_2}(s)}{T_s(s)} = \frac{Z_{R_1} \parallel Z_{R_2}}{(Z_{R_1} \parallel Z_{R_2}) + Z_C} \tag{1}$$

$$= \frac{\frac{R_1 R_2}{R_1+R_2}}{\frac{R_1 R_2}{R_1+R_2} + \frac{1}{Cs}} \tag{2}$$

$$= \frac{R_1 R_2 Cs}{R_1 R_2 Cs + R_1 + R_2}. \tag{3}$$

b. The input impedance Z_{in} is the impedance of all the elements combined in series and parallel; that is,

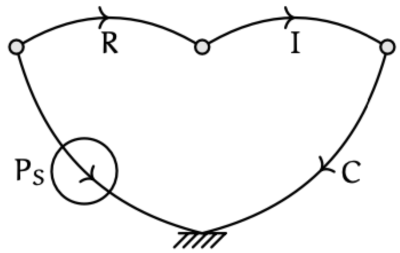
$$Z_{in} = (Z_{R_1} \parallel Z_{R_2}) + Z_C \tag{4}$$

$$= \frac{R_1 R_2}{R_1 + R_2} + \frac{1}{Cs} \tag{5}$$

$$= \frac{R_1 R_2 Cs + R_1 + R_2}{(R_1 + R_2)Cs}. \tag{6}$$

Exercise 13.2 granite

Use the linear graph below of a fluid system to **(a)** derive the transfer function $P_C(s)/P_S(s)$, where P_S is the input pressure and P_C is the pressure across the fluid capacitance C. *Use impedance methods and a divider rule is highly recommended*. (Simplify the transfer function.) And **(b)** derive the input impedance the input P_S drives. (Don't simplify the expression.)



Solution

a. This is a straightforward across-variable divider:

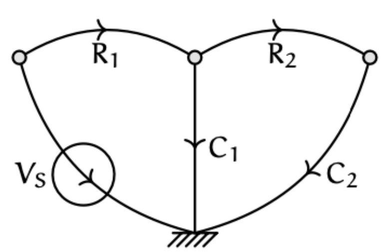
$$\begin{aligned}\frac{P_C(s)}{P_S(s)} &= \frac{Z_C}{Z_I + Z_R + Z_C} \\ &= \frac{1}{ICs^2 + RCs + 1}.\end{aligned}$$

b. The input impedance for the system consisting of three series elements is

$$Z_R + Z_I + Z_C = R + Is + \frac{1}{Cs}.$$

Exercise 13.3 granted

Use the linear graph below of an electronic system to derive the transfer function $I_{R_1}(s)/V_S(s)$, where V_S is the input voltage and I_{R_1} is the current through the resistor R_1. (Simplify the transfer function.) Use an impedance method. *Hint: a divider method is recommended; without it, use of a computer is recommended.*



Solution

We will use an across-variable (voltage) divider. The elements C_1, R_2, and C_2 can all be combined (in series and parallel) to form an equivalent

impedance

$$\begin{aligned} Z_e &= \frac{1}{\frac{1}{Z_{C_1}} + \frac{1}{Z_{R_2}+Z_{C_2}}} \\ &= \frac{1}{C_1 s + \frac{1}{R_2 + \frac{1}{C_2 s}}} \\ &= \frac{1}{C_1 s + \frac{C_2 s}{R_2 C_2 s + 1}} \\ &= \frac{R_2 C_2 s + 1}{(R_2 C_2 s + 1) C_1 s + C_2 s} \\ &= \frac{R_2 C_2 s + 1}{R_2 C_1 C_2 s^2 + (C_1 + C_2) s}. \end{aligned}$$

The voltage divider rule gives

$$\begin{aligned} \frac{V_{R_1}}{V_S} &= \frac{Z_{R_1}}{Z_{R_1} + Z_e} \\ &= \frac{(R_2 C_1 C_2 s^2 + (C_1 + C_2) s) R_1}{(R_2 C_1 C_2 s^2 + (C_1 + C_2) s) R_1 + R_2 C_2 s + 1} \\ &= \frac{R_1 R_2 C_1 C_2 s^2 + R_1 (C_1 + C_2) s}{R_1 R_2 C_1 C_2 s^2 + R_1 (C_1 + C_2) s + R_2 C_2 s + 1} \\ &= \frac{(R_1 R_2 C_1 C_2) s^2 + R_1 (C_1 + C_2) s}{(R_1 R_2 C_1 C_2) s^2 + (R_1 (C_1 + C_2) + R_2 C_2)\, s + 1}. \end{aligned}$$

But we want I_{R_1}/V_S. The impedance elemental equation for R_1 lets us trade $V_{R_1} \mapsto I_{R_1}$:

$$\begin{aligned} V_{R_1} &= I_{R_1} Z_{R_1} \\ &= R_1 I_{R_1}. \end{aligned}$$

Substituting this into the transfer function above,

$$\begin{aligned} \frac{R_1 I_{R_1}}{V_S} &= \frac{(R_1 R_2 C_1 C_2) s^2 + R_1 (C_1 + C_2) s}{(R_1 R_2 C_1 C_2) s^2 + (R_1 (C_1 + C_2) + R_2 C_2)\, s + 1} \Rightarrow \\ \frac{I_{R_1}}{V_S} &= \frac{(R_2 C_1 C_2) s^2 + (C_1 + C_2) s}{(R_1 R_2 C_1 C_2) s^2 + (R_1 (C_1 + C_2) + R_2 C_2)\, s + 1}. \end{aligned}$$

Exercise 13.5 tableau

Consider an accelerometer that has transfer function

$$G(s) \equiv \frac{V_i(s)}{A(s)} = \frac{K_G\,\omega_{n_G}^2}{s^2 + 2\zeta_G \omega_{n_G} s + \omega_{n_G}^2}, \tag{6}$$

where

- A is the input acceleration in $\mathrm{m/s^2}$,
- V_i is the output voltage in V,
- $K_G = 0.1\ \mathrm{V/(m/s^2)}$ is the gain,
- $\omega_{n_G} = 3000\ \mathrm{rad/s}$ is the natural frequency, and
- $\zeta_G = 0.2$ is the damping ratio.

Perform a frequency domain analysis as follows.

a. Generate a Bode plot of $G(s)$.
b. At DC ($\omega = 0\ \mathrm{rad/s}$), compute the *magnitude* and *phase* of the frequency response function of the accelerometer.

Suppose there is a sinusoidal systematic noise signal at the input, with amplitude $a_{\text{noise}} = 1\ \mathrm{m/s^2}$ and frequency $\omega_{\text{noise}} = 2900\ \mathrm{rad/s}$.[1]

c. Assuming there is only noise input, at the noise frequency ω_{noise}, compute the *amplitude* and *phase* of the voltage V_i at the output of the accelerometer. Why is the amplitude higher than it would have been at DC (use your Bode plot from Item a. to justify your answer).

To mitigate the systematic noise, we add a filter with transfer function $H(s)$ to the output of the accelerometer, as shown in Fig. imp.3.

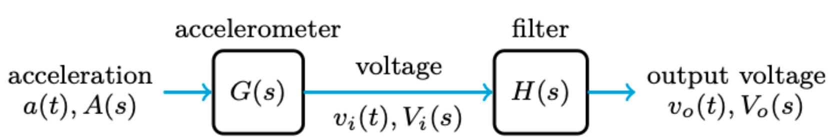


Figure imp.3: Accelerometer and filter block diagram.

[1] Assume the input phase is zero.

By definition,

$$H(s) \equiv \frac{V_o(s)}{V_i(s)}. \tag{7}$$

Assume the filter and accelerometer *do not* dynamically load each other. The filter circuit diagram is shown in Fig. imp.4.

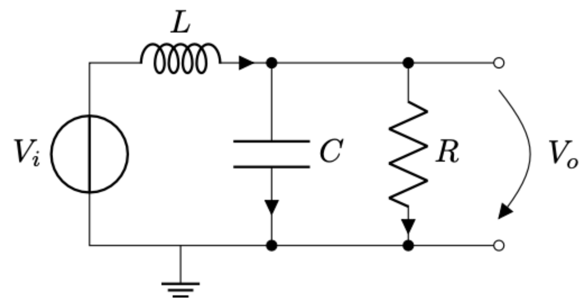


Figure imp.4: Filter circuit.

d. Draw a linear graph model of the filter circuit.
e. Use *impedance methods* to derive the transfer function $H(s)$ in terms of the circuit element parameters R, L, and C.
f. Find the filter's natural frequency ω_{n_H} and damping ratio ζ_H.[2]
g. Let $C = 0.001$ F. Design the filter by choosing R and L such that

$$\zeta_H = 1 \quad \text{and} \quad \omega_{n_H} = 1000 \text{ rad/s}. \tag{8}$$

h. Find the transfer function

$$\frac{V_o(s)}{A(s)} \tag{9}$$

with all parameters substituted. Simplify.
i. Generate a Bode plot for $V_o(s)/A(s)$.
j. Using the Bode plot of Item i., explain why we should expect the output from the systematic noise at ω_{noise} to be improved.
k. From the transfer function $V_o(s)/A(s)$, at the noise frequency ω_{noise}, compute the *amplitude* and *phase* of the output voltage V_o.

[2]Be cautious to make the denominator have the proper standard form $s^2 + 2\zeta_H\omega_{n_H} s + \omega_{n_H}^2$.

l. Compare the result from Item k. to the unfiltered voltage in Item c. by finding the ratio of the filtered amplitude over then unfiltered amplitude.
m. How could you augment the filter design to further reduce the systematic noise?

Solution

a: Bode plot

First, define the system with

```
wn = 3000; % rad/s
z = .2;
K_G = 0.1; % V/(m/s^2)
G = tf([K_G*wn^2],[1,2*z*wn,wn^2])
```

```
G =

         900000
  --------------------
  s^2 + 1200 s + 9e06

Continuous-time transfer function.
```

Now we can generate the Bode plot, shown in Fig. imp.5, with

```
figure;
bode(G);
grid on;
```

b: DC gain and phase

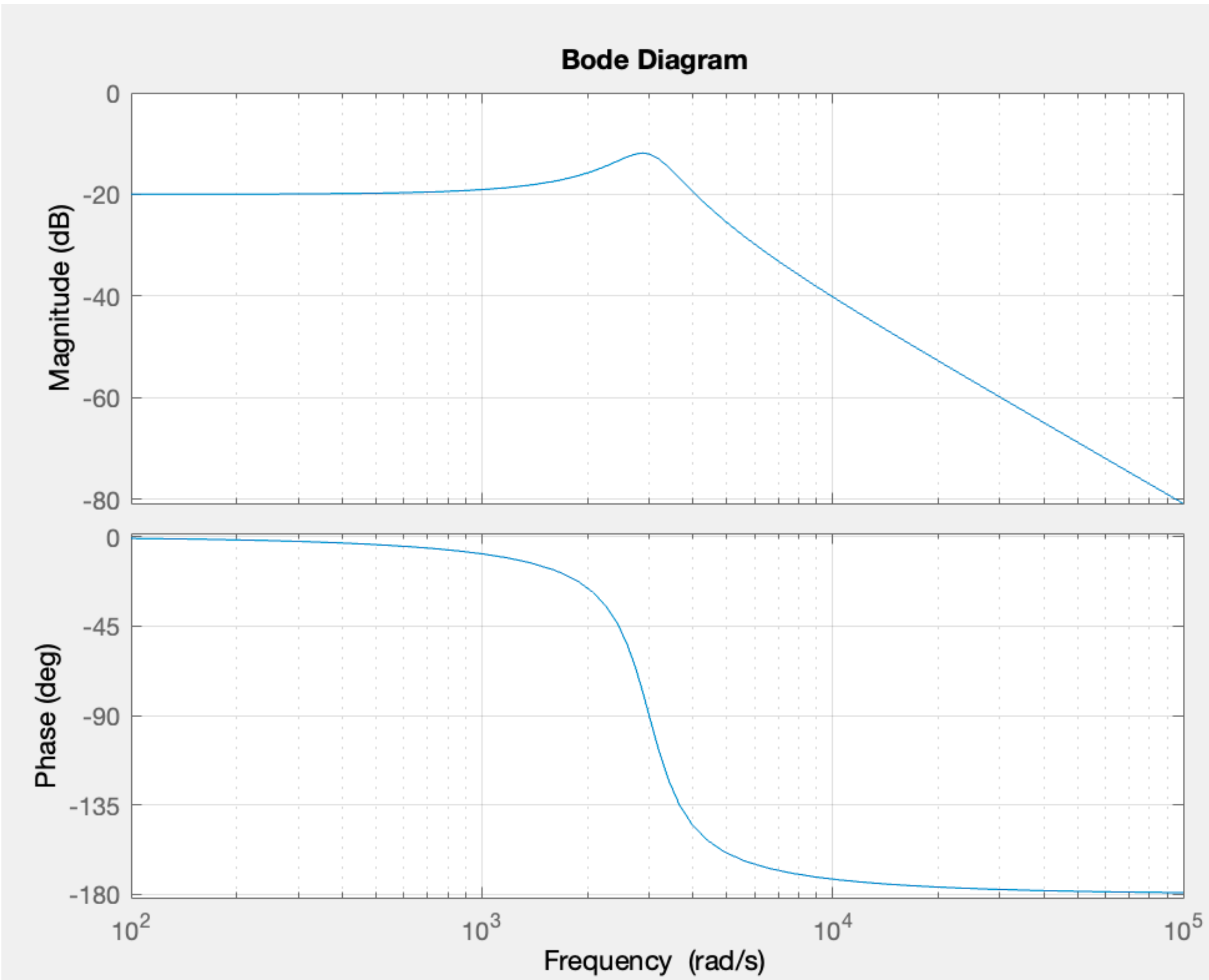
At DC ($\omega = 0$ rad/s),

```
G_DC = freqresp(G,0); % complex FRF at DC
mag_DC = abs(G_DC); % magnitude
pha_DC = angle(G_DC); % phase
```

```
fprintf("DC magnitude = %.4g\n",mag_DC)
fprintf("DC phase = %.4g rad",pha_DC)
```

```
DC magnitude = 0.1
DC phase = 0 rad
```

These are the accelerometer DC gain and phase.



c: Noise output

```
wnoise = 2900; % rad/s ... noise frequency
anoise = 1; % m/s^2 ... noise amplitude
```

The noise output can be computed as

```
Gnoise = freqresp(G,wnoise); % FRF at noise freq.
Vinoise_mag = abs(Gnoise)*anoise; % multiplies
Vinoise_pha = 0 + angle(Gnoise);
fprintf("Noise |Vo| = %.4g V\n",Vinoise_mag)
fprintf("Noise  Vo = %.4g deg",rad2deg(Vinoise_pha))
```

```
Noise |Vo| = 0.255 V
Noise  Vo = -80.38 deg
```

The magnitude of the noise would have been higher at DC because the Bode plot has a peak near the noise frequency.

d: Linear graph model of the filter

The linear graph model is shown in Fig. imp.6.

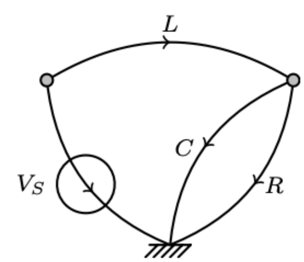


Figure imp.6: Linear graph model.

e: Filter transfer function H(s)

```
syms s
syms R L C
syms ZR ZL ZC
assume(R>=0);
assume(L>=0);
assume(C>=0);
```

```
imp.ZR = R;
imp.ZL = L*s;
imp.ZC = 1/(C*s);
```

```
Zo = 1/(1/ZC + 1/ZR);
HZ = simplify( ...
    Zo/(Zo+ZL) ...
)
H = simplify( ...
    subs(HZ,imp) ...
)
```

$$HZ = \frac{ZC\,ZR}{ZC\,ZL+ZC\,ZR+ZL\,ZR}$$
$$H = \frac{R}{C\,L\,R\,s^2+L\,s+R}$$

f: Filter natural freq. and damping ratio

Use the denominator of the transfer function, like

```
[Hn,Hd] = numden(H);
co = coeffs(Hd,s);
a0 = co(1)/co(3);
a1 = co(2)/co(3);
wnf = sqrt(a0)
znf = simplify( ...
    a1/(2*wnf) ...
)
Kf = limit(H,s,0)
```

$$wnf = \sqrt{\frac{1}{C\,L}}$$
$$znf = \frac{\sqrt{L}}{2\,\sqrt{C}\,R}$$
$$Kf = 1$$

g: Filter design

```
psol = solve([znf == 1,wnf == 1000,C == 1000e-6],[R,L,C]);
R_ = double(psol.R);
```

```
L_ = double(psol.L);
C_ = double(psol.C);
fprintf("R = %.4g Ω\nL = %.4g H\nC = %.4g F",R_,L_,C_)
```

```
R = 0.5 Ω
L = 0.001 H
C = 0.001 F
```

h: Transfer function V_o/A

Since the systems do not load each other, the combined transfer function $V_o(s)/A(s)$ is simply

$$\frac{V_o(s)}{A(s)} = G(s)H(s). \tag{10}$$

Substitute in parameter values and convert the symbolic transfer function for the filter $H(s)$ into a Matlab `tf` object:

```
H_ = simplify(subs(H,psol))
H_ = sym_to_tf(H_,s) % using matlab-rico
```

$$H_= \frac{1000000}{(s+1000)^2}$$

```
H_ =

         1e06
  -------------------
  s^2 + 2000 s + 1e06

Continuous-time transfer function.
```

Now we can find the combined transfer function

```
Vo_A = G*H_
```

```
Vo_A =

                         9e11
```

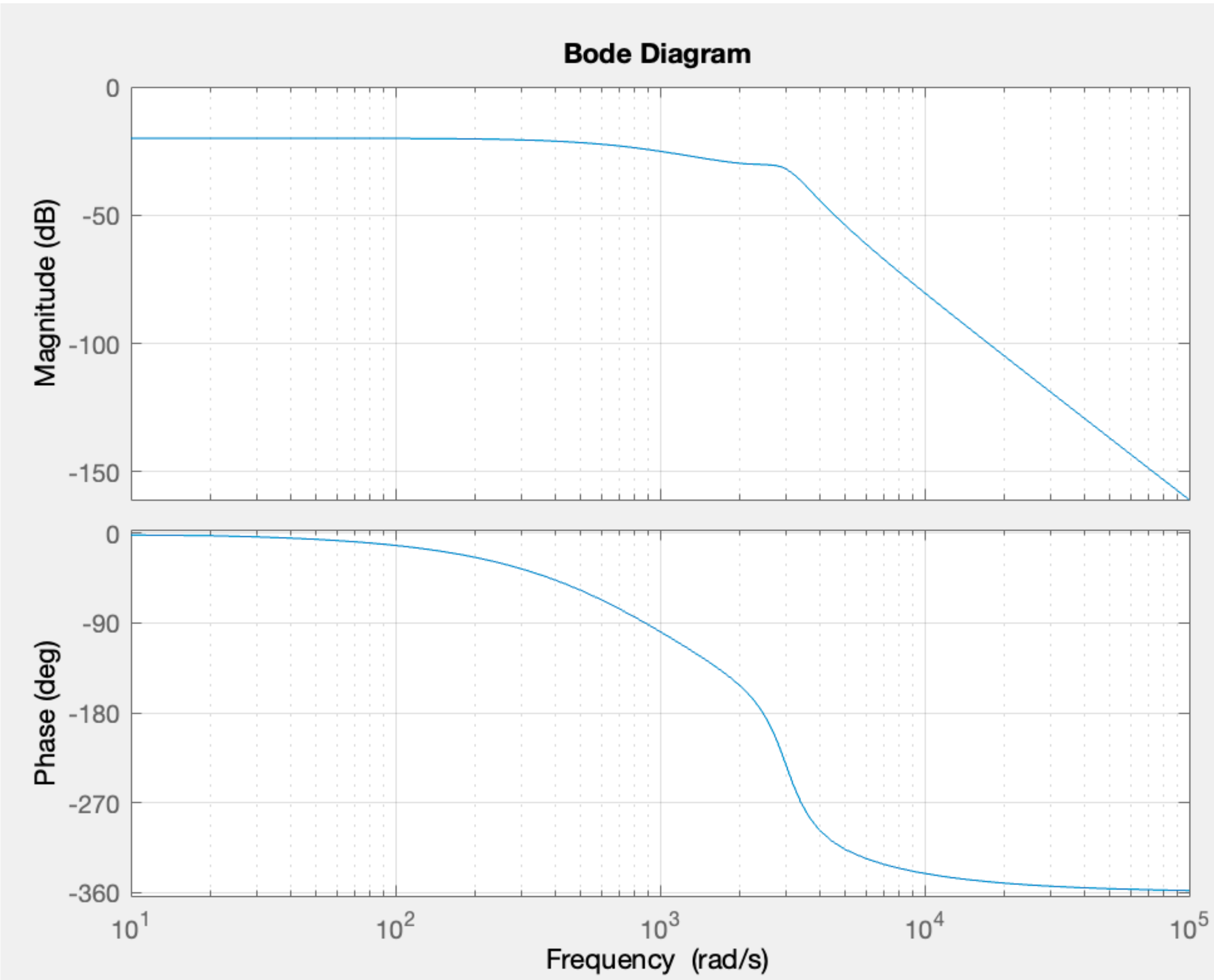
```
  ----------------------------------------------
  s^4 + 3200 s^3 + 1.24e07 s^2 + 1.92e10 s + 9e12

Continuous-time transfer function.
```

i: Bode plot for V_o/A

Now we can see the new Bode plot, shown in Fig. imp.7, with

```
figure;
bode(Vo_A);
grid on;
```



j: Noise improvement with filter

We should expect the noise output to be lower because, as shown in Fig. imp.7, the transfer function V_o/A has lower magnitude at the noise frequency than does the transfer function G (with its Bode plot shown in Fig. imp.5).

k: Filtered noise output

```
GHnoise = freqresp(Vo_A,wnoise); % FRF
Vonoise_mag = abs(GHnoise)*anoise;
Vonoise_pha = 0 + angle(GHnoise) - 2*pi;
fprintf("Noise |Vo| = %.4g V\n",Vonoise_mag)
fprintf("Noise  Vo = %.4g deg",rad2deg(Vonoise_pha))
```

```
Noise |Vo| = 0.0271 V
Noise  Vo = -222.3 deg
```

```
fprintf("Noise ratio = %.4g", ...
    Vonoise_mag/Vinoise_mag ...
)
```

```
Noise ratio = 0.1063
```

This means the filtered noise magnitude is about 1/10 that of the unfiltered noise.

m: Augmenting the filter

We could further reduce the noise by decreasing the filter's natural frequency, which we set at 1000 rad/s. The lower this frequency, the more aggressive will be the filter. The downside is that the dynamic response of the measurement will be slower (a smaller bandwidth yields slower responses).

Problem 13.5

(a)

$$Z(s) = \frac{V(s)}{F(s)} = \frac{s}{K} + \frac{1}{ms} = \frac{ms^2 + K}{mKs}$$

(b)

$$Z(s) = \frac{P(s)}{Q(s)} = \frac{1}{sC_f + \frac{1}{sI_f + R_1 + R_2}} = \frac{I_f s + (R_1 + R_2)}{C I_f s^2 + C\,(R_1 + R_2)\,s + 1}$$

(c)

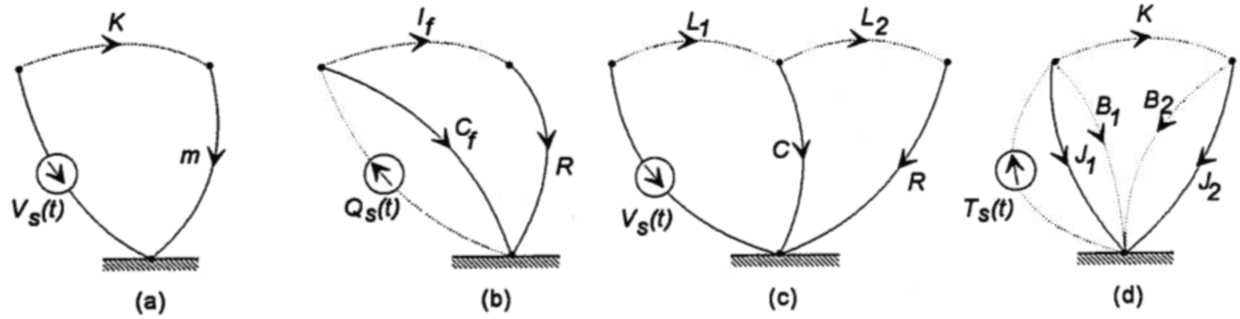
$$\begin{aligned} Z(s) &= \frac{V(s)}{I(s)} = sL_1 + \frac{1}{sC + \frac{1}{sL_2 + R}} \\ &= sL_1 + \frac{sL_2 + R}{sC(sL_2 + R) + 1} \\ &= \frac{L_1 L_2 C s^3 + L_1 C s^2 + (L_1 + L_2)s + R}{L_2 C s^2 + RCs + 1} \end{aligned}$$

(d)

$$Z(s) = \frac{J_2 s^2 + B_2 s + K}{J_1 J_2 s^3 + (B_1 J_2 + B_2 J_1)\, s^2 + (B_1 B_2 + J_1 K)\, s + B_1 K}$$

Problem 13.9

Using the methods of Sec. 13.4:



(a) The tree generates the following constraint equations

missing damper in part a

$$\begin{aligned} v_K + v_m - V_(s) &= 0 \\ F_K - F_m &= 0 \end{aligned}$$

When the impedance relationships for the secondary variables are substituted, these equations may be written in matrix form:

$$\begin{bmatrix} Z_K & 1 \\ 1 & -Y_m \end{bmatrix} \begin{bmatrix} F_K \\ v_m \end{bmatrix} = \begin{bmatrix} V_s \\ 0 \end{bmatrix}$$

which may be solved (Cramer's Rule) to give

$$H(s) = \frac{v_m(s)}{V_s(s)} = \frac{K}{ms^2 + K}$$

The differential equations is

$$m\frac{d^2 v_m}{dt^2} + Kv_m = KV_s(t)$$

(b) Let $R = R_1 + R_2$, and note that $Q_{R_1} = Q_R = Q_{I_f}$. The tree generates the following constraint equations

$$\begin{aligned} Q_R - Q_{I_f} &= 0 \\ Q_s - Q_{C_f} - Q_{I_f} &= 0 \\ P_{I_f} + P_R - P_{C_f} &= 0 \end{aligned}$$

When the impedance relationships for the secondary variables are substituted, these equations may be written in matrix form:

$$\begin{bmatrix} -1 & 0 & Y_R \\ 1 & Y_{C_f} & 0 \\ Z_{I_f} & -1 & 1 \end{bmatrix} \begin{bmatrix} Q_{I_f} \\ P_{C_f} \\ P_R \end{bmatrix} = \begin{bmatrix} 0 \\ Q_s \\ 0 \\ 0 \end{bmatrix}$$

which may be solved (Cramer's Rule) to give

$$H(s) = \frac{Q_{R_2}(s)}{Q_s(s)} = \frac{Q_{I_f}(s)}{Q_s(s)} = \frac{1}{I_f C_f s^2 + (R_1 + R_2) C_f s + 1}$$

The differential equations is

$$I_f C_f \frac{d^2 Q_{R_2}}{dt^2} + (R_1 + R_2) C_f \frac{dQ_{R_2}}{dt} + Q_{R_2} = Q_s(t)$$

(c) The tree generates the following constraint equations

$$\begin{aligned} V_{L_2} + V_R - v_C &= 0 \\ v_{L_1} + v_C - V_s(t) &= 0 \\ i_{L_2} - I_R &= 0 \\ i_{L_1} - i_C - I_{L_2} &= 0 \end{aligned}$$

When the impedance relationships for the secondary variables are substituted, these equations may be written in matrix form:

$$\begin{bmatrix} 1 & -1 & 0 & Z_{L_2} \\ 0 & 1 & Z_{L_1} & 0 \\ -Y_R & 0 & 0 & 1 \\ 0 & -Y_C & 1 & -1 \end{bmatrix} \begin{bmatrix} v_R \\ v_C \\ i_{L_1} \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} 0 \\ V_s(t) \\ 0 \\ 0 \end{bmatrix}$$

which may be solved (Cramer's Rule) to give

$$H(s) = \frac{v_R(s)}{V_s(s)} = \frac{R}{L_1 L_1 C s^3 + L_1 C R s^2 + (L_1 + L_2) s + R}$$

The differential equations is

$$L_1 L_2 C \frac{d^3 v_{R_2}}{dt^3} + L_1 C R \frac{d^2 v_{R_2}}{dt^2} + (L_1 + L_2) \frac{dv_{R_2}}{dt} + R v_{R_2} = R V_s(t)$$

(d) The tree generates the following constraint equations

$$\begin{aligned} \Omega_{J_1} - \Omega_{B_1} &= 0 \\ \Omega_{J_2} - \Omega_{B_2} &= 0 \\ T_s - T_{B_1} - T_{J_1} - T_K &= 0 \\ T_K - T_{B_2} - T_{J_1} &= 0 \\ \Omega_K + \Omega_{J_2} - \Omega_{J_1} &= 0 \end{aligned}$$

When the impedance relationships for the secondary variables are substituted, these equations may be written in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & -Y_{B_1} & 0 \\ 0 & 1 & 0 & 0 & -Y_{B_2} \\ Y_{J_1} & 0 & 1 & 1 & 0 \\ 0 & -Y_{J_2} & 1 & 0 & -1 \\ -1 & 1 & Z_K & 0 & 0 \end{bmatrix} \begin{bmatrix} \Omega_{J_1} \\ \Omega_{J_1} \\ T_K \\ T_{B_1} \\ T_{B_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ T_s \\ 0 \\ 0 \end{bmatrix}$$

which may be solved to give

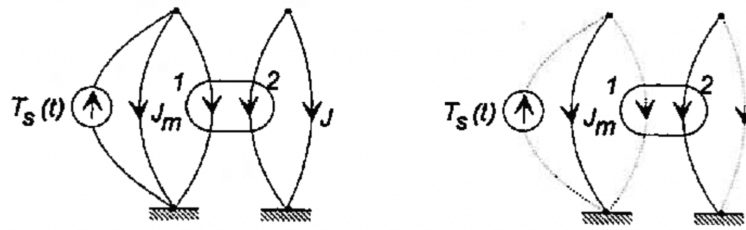
$$H(s) = \frac{\Omega_{J_2}(s)}{T_s(s)} = \frac{K}{J_1 J_2 s^3 + (J_1 B_2 + J_2 B_1)s^2 + (K(J_1 + J_2) + B_1 B_2)s + K(B_1 + B_2)}$$

The differential equations is

$$J_1 J_2 \frac{d^3\Omega_{J_2}}{dt^3} + (J_1 B_2 + J_2 B_1)\frac{d^2\Omega_{J_2}}{dt^2} + (K(J_1 + J_2) + B_1 B_2)\frac{d\Omega_{J_2}}{dt} + K(B_1 + B_2)\Omega_{J_2} = KT_s(t)$$

Problem 13.18

(a) The linear graph is shown below:



(b) The transformer relationship for the gears is

$$
\begin{aligned}
\Omega_1 &= -\frac{1}{N}\Omega_2 \\
T_1 &= NT_2
\end{aligned}
$$

From the tree the following equations may be derived

$$
\begin{bmatrix} sJm & 1 & 0 & 0 \\ 0 & -1/N & 0 & -1 \\ -1 & 0 & -1/N & 0 \\ 0 & 0 & -1 & 1/sJ \end{bmatrix}
\begin{bmatrix} \Omega_{J_m} \\ T_1 \\ \Omega_2 \\ T_J \end{bmatrix}
=
\begin{bmatrix} T_s \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

which may be solved (Cramer's Rule) to give

$$
Z(s) = \frac{\Omega_J(s)}{T_s(s)} = \frac{1}{s(J + N^2 J)}
$$

(c) The effective moment of inertia seen by the motor is $J_m + N^2 J$, that is the flywheel inertia reflected through the gear train is scaled by a factor N^2.