Murray Problem 3.10: Feedforward LQR with Integral Action

Source Filename: /main.py

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Begin by importing the necessary libraries and modules as follows:

```
import numpy as np
import matplotlib.pyplot as plt
import control
```

Define the linear system dynamics as follows:

```
A = np.array([[0, 2], [-1, -0.1]])
B = np.array([[0], [1]])
C = np.eye(2)
D = np.zeros((2,1))
sys = control.ss(A, B, C, D)
```

Extract the number of states, inputs, and outputs as follows:

```
n = A.shape[0]  # Number of states
m = B.shape[1]  # Number of inputs
r = C.shape[0]  # Number of outputs
```

Compute the equilibrium point as follows:

```
xd = np.array([[1], [0]])  # Desired equil. state (given)
ud = np.array([[1]])  # Desired equil. input (from 0 = A*xd + B*ud)
    # This could be computed automatically, but it's a little involved
    # because in general the system is overdetermined.
if (A @ xd + B @ ud != np.zeros_like(xd)).any():  # Check validity
    raise ValueError('The desired equilibrium point is not valid.')
```

Compute an LQR controller as follows:

```
Q = np.eye(n)  # State cost matrix
R = np.eye(m)  # Input cost matrix
K, _, _ = control.lqr(A, B, Q, R)
```

We can compute the closed-loop system dynamics with a feedforward input term as follows:

```
ctrl, clsys = control.create_statefbk_iosystem(sys, K)
print(clsys)
```

```
<LinearICSystem>: sys[0]_sys[1]
Inputs (3): ['xd[0]', 'xd[1]', 'ud[0]']
Outputs (3): ['y[0]', 'y[1]', 'u[0]']
States (2): ['sys[0]_x[0]', 'sys[0]_x[1]']

A = [[ 0.          2.        ]
     [-1.41421356 -1.6330506 ]]

B = [[0.         0.         0.        ]
     [0.41421356 1.5330506  1.        ]]

C = [[ 1.          0.        ]
     [ 0.          1.        ]
     [-0.41421356 -1.5330506 ]]

D = [[0.         0.         0.        ]
     [0.         0.         0.        ]
     [0.41421356 1.5330506  1.        ]]
```

The first two inputs to the closed-loop system are the desired state, and the third input is the feedforward input. The response of the closed-loop system to an input of

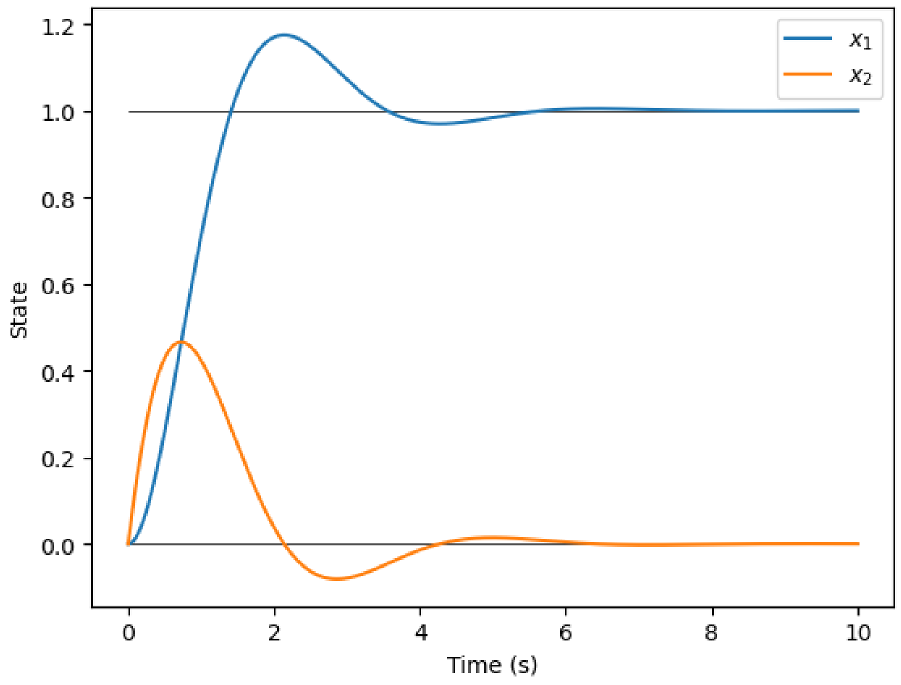
$$\begin{bmatrix} x_d \\ u_d \end{bmatrix}$$

can be simulated as follows:

```
t = np.linspace(0, 10, 1000)  # Simulation time
input_d = np.vstack([xd, ud]) @ \
    np.array([np.ones_like(t)])  # Desired state and input over time
_, output = control.forced_response(clsys, T=t, U=input_d)  # Simulate
```

Plot the response of the closed-loop system as follows:

```
fig, ax = plt.subplots()
ax.plot(t, input_d[0], 'k', linewidth=0.5)
ax.plot(t, input_d[1], 'k', linewidth=0.5)
ax.plot(t, output[0], label='$x_1$')
ax.plot(t, output[1], label='$x_2$')
ax.set_xlabel('Time (s)')
ax.set_ylabel('State')
ax.legend()
plt.draw()
```

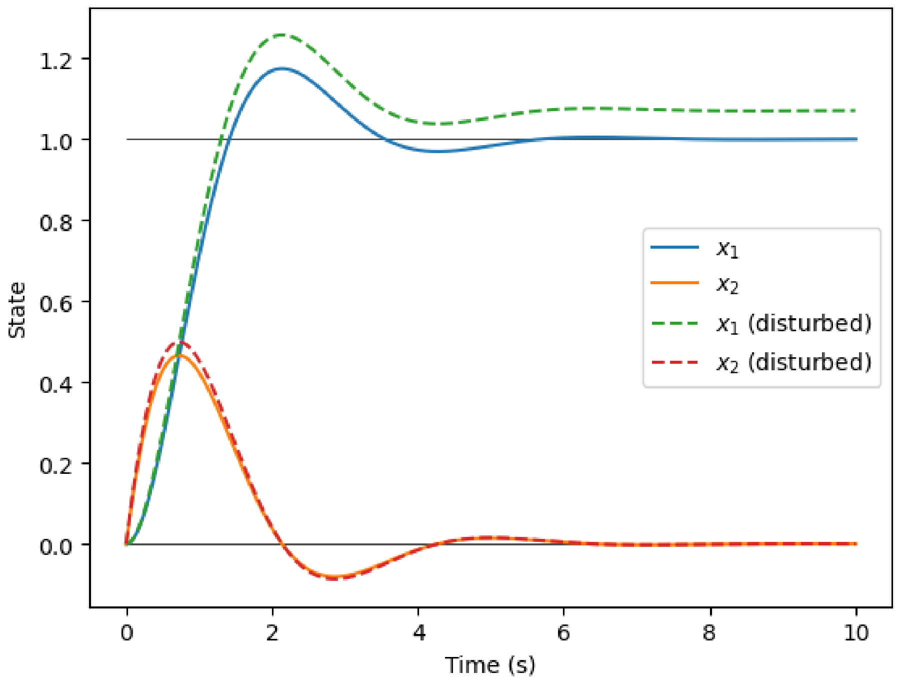


Now supposed there is an input disturbance $d = 0.1$ acting on the input to the plant. We can simulate the response of the closed-loop system to this disturbance as follows:

```
d = 0.1  # Input disturbance
input_d = np.vstack([xd, ud+d]) @ \
    np.array([np.ones_like(t)])  # Disturbed state and input over time
_, output_d = control.forced_response(clsys, T=t, U=input_d)  # Simulate
```

Plot the response of the closed-loop system to the disturbance along with the undisturbed response as follows:

```
ax.plot(t, output_d[0], '--', label='$x_1$ (disturbed)')
ax.plot(t, output_d[1], '--', label='$x_2$ (disturbed)')
ax.legend()
plt.draw()
fig
```



```
<Figure size 640x480 with 0 Axes>
```

The response of the closed-loop system to the disturbance is shown in the dashed lines. The disturbance causes the first state to deviate from the desired value. The feedback controller cannot reject this disturbance because it enters the system directly at the input. This is different than when the disturbance enters the system state or output, where the feedback controller can reject it.

One approach to rejecting the disturbance is to add integral action to the controller. This can be done by augmenting the state-space system with an integrator (see Astrom 2021, p. 8-27 and Murray 2023, pp. 3-18 through 3-19). The integral action is added to the controller as follows:

$$u = u_d - K_r(x - x_d) - K_i \int_0^t C(x - x_d)dt$$

where K_r is the state feedback gain, K_i is the integral gain, and C determines which states receive integral action (and how). To realize this, the state vector is augmented as follows:

$$x \mapsto \xi = \begin{bmatrix} x \\ z \end{bmatrix},$$

where z is the integral of the error state $x - x_d$. The dynamics of the augmented system are given by

$$A \mapsto \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \text{and} \quad B \mapsto \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

The desired equilibrium point is augmented as

$$x_d \mapsto \begin{bmatrix} x_d \\ 0 \end{bmatrix}$$

because the integral state should be zero at the desired equilibrium point.

The `create_statefbk_iosystem` function can be used to create a state-space system with integral action by selecting a matrix C, augmenting A and B as shown above and augmenting K_r with an integral gain K_i; that is,

$$K_r \mapsto \begin{bmatrix} K_r \\ K_i \end{bmatrix}.$$

The greater the values of K_i, the faster the integral action will respond to errors. However, large values of K_i can affect system performance and stability. We will select K_i by including it in the augmented LQR design. Increasing the Q matrix values for the integral state will increase the integral gain and make the integral action faster.

The integral action can be added to the controller as follows:

```
C_int = np.array([[1, 0]])  # Select the state to get integrator
A_aug = np.block([[A, np.zeros((n, 1))], [C_int, 0]])
B_aug = np.vstack([B, np.zeros((1, m))])
Q_aug = np.eye(n+1)  # State+integral cost matrix
K_aug, _, _ = control.lqr(A_aug, B_aug, Q_aug, R)
ctrl_int, clsys_int = control.create_statefbk_iosystem(
    sys, K_aug, integral_action=C_int
)
print(clsys_int)
```

```
<LinearICSystem>: sys[0]_sys[3]
Inputs (3): ['xd[0]', 'xd[1]', 'ud[0]']
Outputs (3): ['y[0]', 'y[1]', 'u[0]']
States (3): ['sys[0]_x[0]', 'sys[0]_x[1]', 'sys[3]_x[0]']

A = [[ 0.          2.          0.        ]
     [-2.07460956 -2.30400483 -1.        ]
     [ 1.          0.          0.        ]]

B = [[ 0.          0.          0.        ]
     [ 1.07460956  2.20400483  1.        ]
     [-1.          0.          0.        ]]

C = [[ 1.          0.          0.        ]
     [ 0.          1.          0.        ]
     [-1.07460956 -2.20400483 -1.        ]]
```

```
D = [[0.         0.         0.         ]
     [0.         0.         0.         ]
     [1.07460956 2.20400483 1.         ]]
```

The response of the closed-loop system with integral action to a disturbed input of

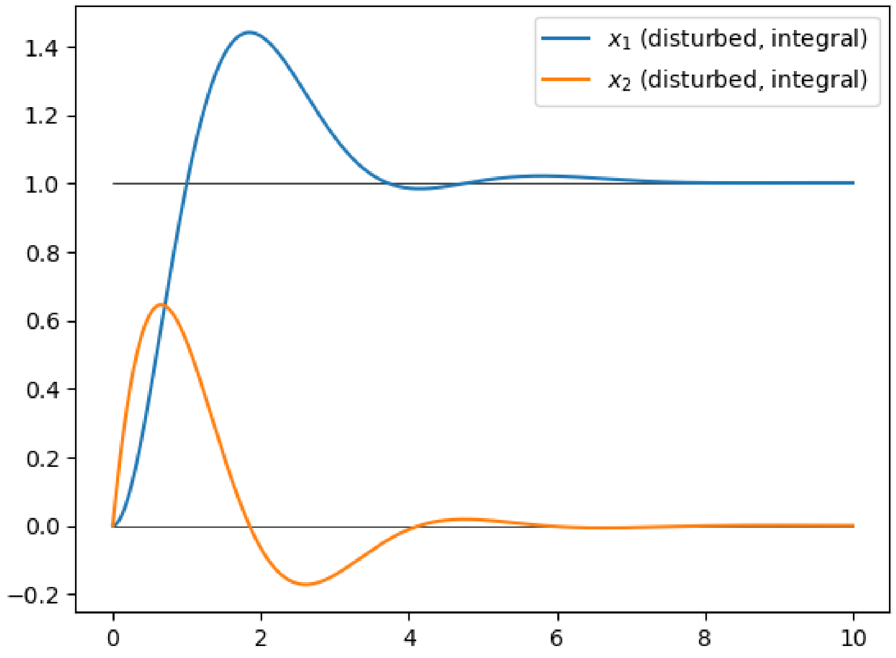
$$\begin{bmatrix} x_d \\ u_d + d \end{bmatrix}$$

can be simulated as follows:

```
_, output_int = control.forced_response(clsys_int, T=t, U=input_d)  # Simulate
```

Plot the response of the closed-loop system with integral action to the disturbance along with the undisturbed response as follows:

```
fig, ax = plt.subplots()
ax.plot(t, input_d[0], 'k', linewidth=0.5)
ax.plot(t, input_d[1], 'k', linewidth=0.5)
ax.plot(t, output_int[0], label='$x_1$ (disturbed, integral)')
ax.plot(t, output_int[1], label='$x_2$ (disturbed, integral)')
ax.legend()
plt.draw()
```



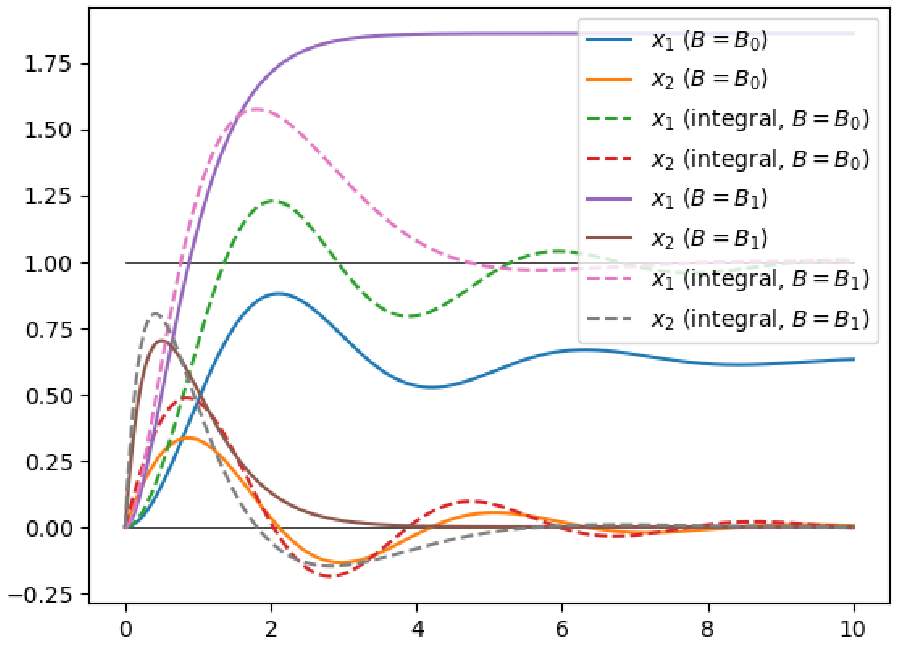
The steady-state error in the first state is eliminated by the integral action. However, note that the overshoot is increased.

Now consider two different variations of the B matrix. For each, we will try the original controller and the controller with integral action.

```
Bs = [
    np.array([[0], [0.5]]),
    np.array([[0], [2.5]])
]
outputs = []
outputs_int = []
for B in Bs:
    sys = control.ss(A, B, C, D)
    ctrl, clsys = control.create_statefbk_iosystem(sys, K)
    ctrl_int, clsys_int = control.create_statefbk_iosystem(
        sys, K_aug, integral_action=C_int
    )
    _, output = control.forced_response(clsys, T=t, U=input_d)
    _, output_int = control.forced_response(clsys_int, T=t, U=input_d)
    outputs.append(output)
    outputs_int.append(output_int)
```

Plot the responses of the closed-loop systems with and without integral action for the two different B matrices as follows:

```
fig, ax = plt.subplots()
ax.plot(t, input_d[0], 'k', linewidth=0.5)
ax.plot(t, input_d[1], 'k', linewidth=0.5)
for i in range(len(Bs)):
    ax.plot(t, outputs[i][0], label=f'$x_1$ ($B = B_{i}$)')
    ax.plot(t, outputs[i][1], label=f'$x_2$ ($B = B_{i}$)')
    ax.plot(t, outputs_int[i][0], '--', label=f'$x_1$ (integral, $B = B_{i}$)')
    ax.plot(t, outputs_int[i][1], '--', label=f'$x_2$ (integral, $B = B_{i}$)')
ax.legend()
plt.show()
```



The response of the closed-loop system with integral action is shown in the dashed lines. The integral action eliminates the steady-state error in the first state for both B matrices. The transient performance is affected, but the system remains stable. This demonstrates the effectiveness of integral action in rejecting disturbances that enter the system directly at the input or when the system model is uncertain. Note that in the latter case, without feedforward control, the steady-state error would likely have been eliminated. The reason it is not here is that we are using feedforward control based on an incorrect model. This is a common issue in practice, and integral action can help to mitigate it.

The value of the feedforward control is that it can be used to more accurately track a desired trajectory. In this case, we are providing a static desired equilibrium point, which is not a feasible trajectory, so it cannot be tracked exactly.

Murray Problem 3.12: Trajectory Optimization of a Thrust-Vectoring Aircraft

Source Filename: /main.py

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In this problem, we will compute optimal trajectories for the PVTOL system using two diferent methods: numerical optimization and exploiting (near) differential flatness. The numerical optimization method allows us to compute an optimal trajectory $(x_d(t), u_d(t))$ that minimizes a cost function J subject to the dynamics of the system. The numerical nature of this approach requires some care in selecting the parameters of the optimization problem, such as the initial guess, the cost function, and the constraints. The `control.optimal` module provides a function `solve_ocp()` that can be used to solve optimal control problems of this form.

Begin by importing the necessary libraries and modules as follows:

```python
import numpy as np
import matplotlib.pyplot as plt
import control
import control.optimal as opt
import control.flatsys as fs
import time
from pvtol import pvtol, plot_results
```

The PVTOL system dynamics and utility functions are defined in the `pvtol` module, found here. We have loaded the dynamics as `pvtol`. Here is some basic information about the system:

```python
print(f"Number of states: {pvtol.nstates}")
print(f"Number of inputs: {pvtol.ninputs}")
print(f"Number of outputs: {pvtol.noutputs}")
```

```
Number of states: 6
Number of inputs: 2
Number of outputs: 6
```

From `pvtol.py`, the state vector x and input vector u are defined as follows:

$$x = \begin{bmatrix} x & y & \theta & \dot{x} & \dot{y} & \dot{\theta} \end{bmatrix}^\top \quad u = \begin{bmatrix} F_1 & F_2 \end{bmatrix}^\top$$

Our first task is to determine the equilibrium state and input that corresponds to a hover at the origin:

```
xeq, ueq = control.find_eqpt(
    pvtol,  # System dynamics
    x0=np.zeros((pvtol.nstates)),  # Initial guess for equilibrium state
    u0=np.zeros((pvtol.ninputs)),  # Initial guess for equilibrium input
    y0=np.zeros((pvtol.noutputs)),  # Initial guess for equilibrium output
    iy=[0, 1],  # Indices of states that should be zero at equilibrium
)
print(f'Equilibrium state: {xeq}')
print(f'Equilibrium input: {ueq}')
```

```
Equilibrium state: [0. 0. 0. 0. 0. 0.]
Equilibrium input: [ 0.  39.2]
```

Set up the optimization problem parameters as follows:

```
x0 = xeq  # Initial state
u0 = ueq  # Initial input
xf = x0 + np.array([1, 0, 0, 0, 0, 0])  # Final state
Q = np.diag([1, 10, 10, 0, 0, 0])  # State cost matrix
R = np.diag([10, 1])  # Input cost matrix
cost = opt.quadratic_cost(pvtol, Q=Q, R=R, x0=xf, u0=u0)  # J
```

The cost function `cost` is constructed with `x0=xf`, the final state, because this is the desired state where the cost should be zero.

Set up the time horizon, time steps, and initial guess for the trajectory, a straight line from `x0` to `xf`, as follows:

```
tf = 5  # Time horizon
nt = 14  # Number of time steps
t = np.linspace(0, tf, nt)  # Time steps
def init_straight(x0, xf, u0, t):
    """Straight-line trajectory with constant input"""
    nt = t.size
    nstates = x0.size
    return np.vstack([
        np.linspace(x0[i], xf[i], nt) for i in range(nstates)
    ]), np.outer(u0, np.ones_like(t))
x_init, u_init = init_straight(x0, xf, u0, t)
```

Solve the optimal control problem as follows:

```
solve_time_start = time.time()
opt1 = opt.solve_ocp(
    pvtol, t, x0, cost, log=True,
    initial_guess=(x_init, u_init),
)
```

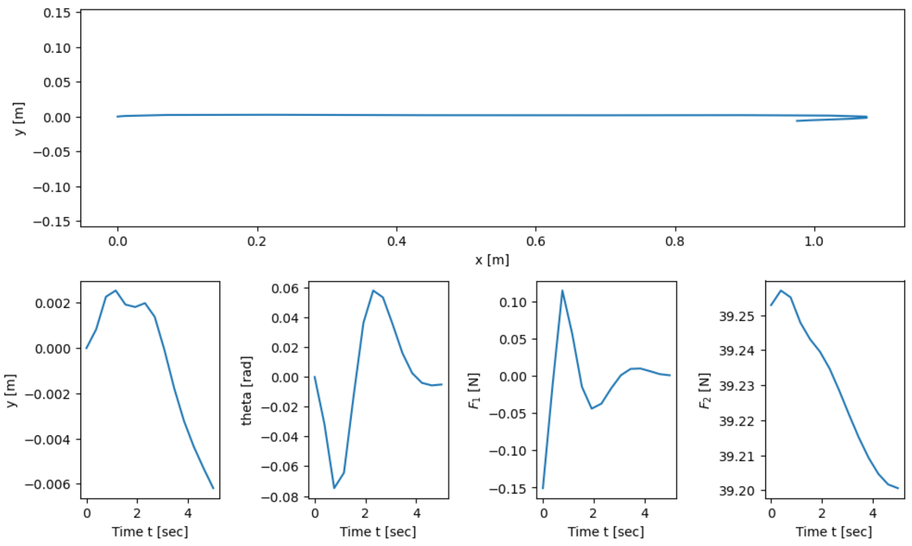
```
solve_time_end = time.time()
print(f"Optimization time: {solve_time_end - solve_time_start:.2f} s")
```

```
Summary statistics:
* Cost function calls: 1821
* Cost function process time: 0.13187099999999852
* System simulations: 0
* Final cost: 1.5053222300852327
Optimization time: 0.42 s
```

Extract the optimal trajectory and plot the results as follows:

```
print(f"Initial position: {opt1.states[0:2, 0]}")
print(f"Final position: {opt1.states[0:2, -1]}")
plot_results(opt1.time, opt1.states, opt1.inputs)
plt.draw()
```

```
Initial position: [0. 0.]
Final position: [ 0.97535942 -0.00619235]
```



We observe that the final value is fairly close to the desired final state. Some overshoot has occurred. The response is not particularly smooth, which is primarily due to the sparsity of the time steps. We could increase the number of time steps to improve the smoothness, but this would also increase the computation time.

Instead, let's try to smooth out the trajectory by using Bezier curves as the parameterization (instead of straight lines between time points).

```
nt2 = 2*nt  # More is OK due to the efficiency of Bezier curves
t2 = np.linspace(0, tf, nt2)
```

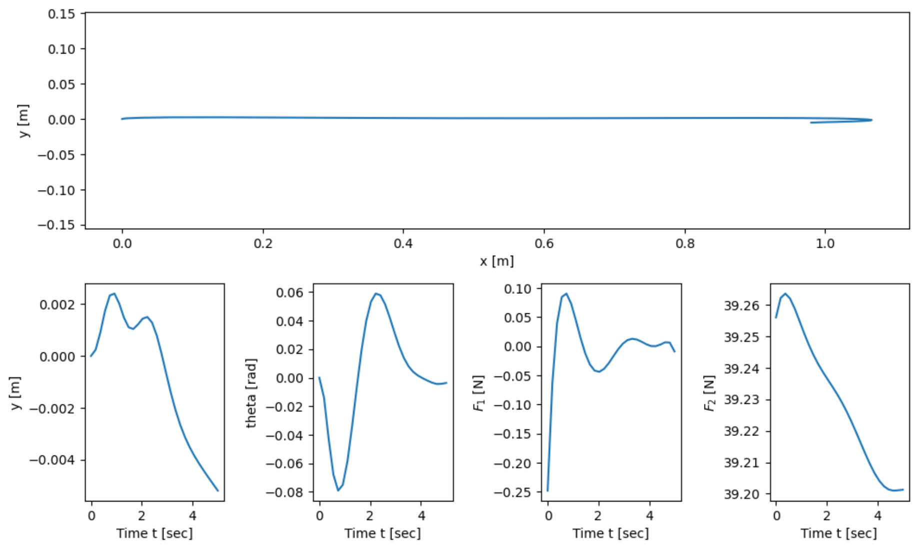
```
x_init2, u_init2 = init_straight(x0, xf, u0, t2)
basis = fs.BezierFamily(8, T=tf)
solve_time_start = time.time()
opt2 = opt.solve_ocp(
    pvtol, t2, x0, cost, log=True,
    initial_guess=(x_init2, u_init2),
    basis=basis,
)
solve_time_end = time.time()
print(f"Optimization time: {solve_time_end - solve_time_start:.2f} s")
```

```
Summary statistics:
* Cost function calls: 5367
* Cost function process time: 4.303219000000034
* System simulations: 0
* Final cost: 1.4445212102307345
Optimization time: 9.74 s
```

Extract the optimal trajectory and plot the results as follows:

```
print(f"Initial position: {opt2.states[0:2, 0]}")
print(f"Final position: {opt2.states[0:2, -1]}")
plot_results(opt2.time, opt2.states, opt2.inputs)
plt.draw()
```

```
Initial position: [0. 0.]
Final position: [ 0.98021887 -0.00520007]
```



The Bezier curve parameterization has smoothed out the trajectory significantly, but the computation time has increased. In some cases, using Bezier curves instead of straight lines with more time steps can be more efficient.

Now create both a trajectory cost and a terminal cost that penalizes the final state and input as follows:

```
cost_traj = opt.quadratic_cost(pvtol, Q=Q/10, R=R, x0=xf, u0=u0)
cost_term = opt.quadratic_cost(pvtol, Q=Q*10, R=R, x0=xf, u0=u0)
```

The trajectory cost `cost_traj` penalizes the trajectory, while the terminal cost `cost_term` penalizes the final state and input. Solve the optimal control problem with the trajectory and terminal costs as follows:

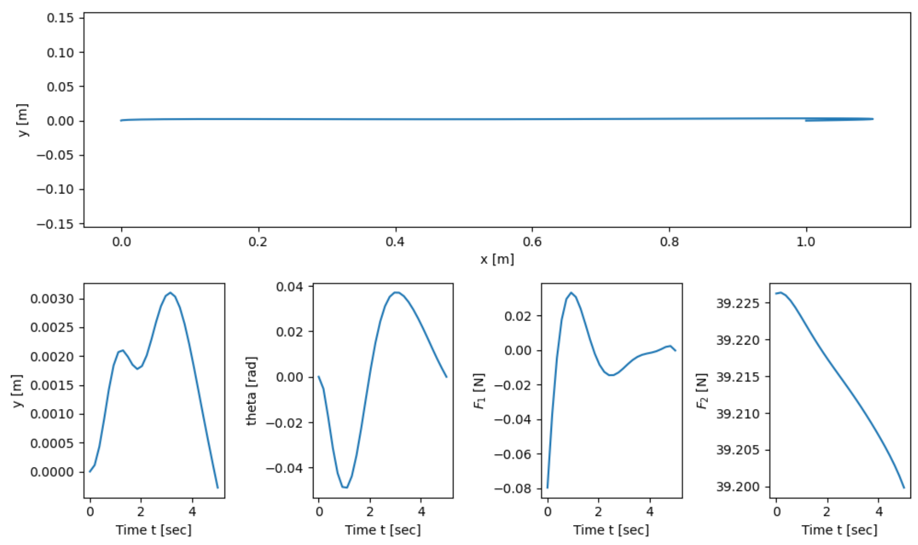
```
solve_time_start = time.time()
opt3 = opt.solve_ocp(
    pvtol, t2, x0, cost_traj, terminal_cost=cost_term,
    initial_guess=(x_init2, u_init2),
    basis=basis,  # Bezier curve basis again
)
solve_time_end = time.time()
print(f"Optimization time: {solve_time_end - solve_time_start:.2f} s")
```

```
Summary statistics:
* Cost function calls: 6293
* System simulations: 0
* Final cost: 0.18210007813464557
Optimization time: 11.35 s
```

Extract the optimal trajectory and plot the results as follows:

```
print(f"Initial position: {opt3.states[0:2, 0]}")
print(f"Final position: {opt3.states[0:2, -1]}")
plot_results(opt3.time, opt3.states, opt3.inputs)
plt.draw()
```

```
Initial position: [0. 0.]
Final position: [ 9.99795695e-01 -2.79186063e-04]
```



The terminal cost has improved the steady-state error.

Now use a terminal *constraint* (not cost) to enforce the final state to be exactly at the desired final state as follows:

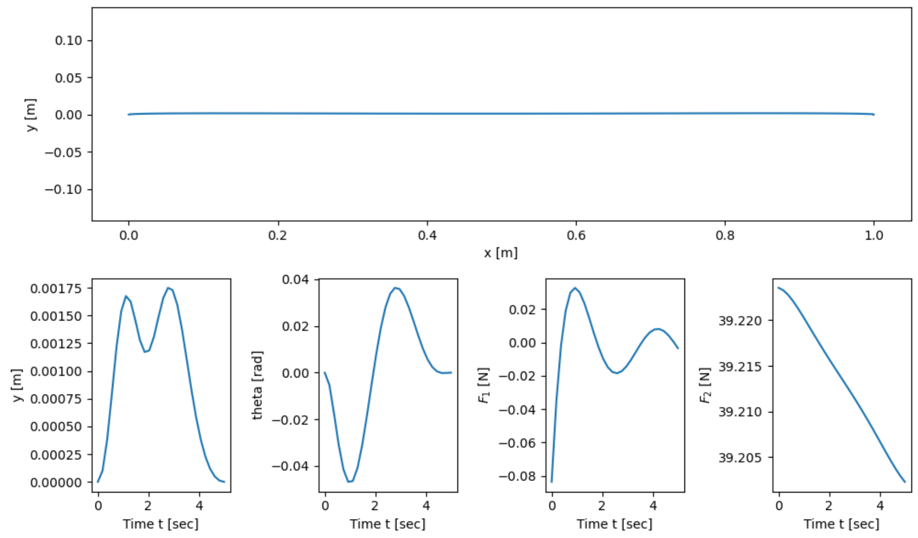
```
con_term = opt.state_range_constraint(pvtol, lb=xf, ub=xf)
solve_time_start = time.time()
opt4 = opt.solve_ocp(
    pvtol, t2, x0, cost_traj, terminal_constraints=con_term,
    initial_guess=(x_init2, u_init2),
    basis=basis,
)
solve_time_end = time.time()
print(f"Optimization time: {solve_time_end - solve_time_start:.2f} s")
```

```
Summary statistics:
* Cost function calls: 2776
* Eqconst calls: 2977
* System simulations: 0
* Final cost: 0.18477758399760782
Optimization time: 7.20 s
```

Extract the optimal trajectory and plot the results as follows:

```
print(f"Initial position: {opt4.states[0:2, 0]}")
print(f"Final position: {opt4.states[0:2, -1]}")
plot_results(opt4.time, opt4.states, opt4.inputs)
plt.draw()
```

```
Initial position: [0. 0.]
Final position: [1. 0.]
```



The terminal constraint has forced the final state to be exactly at the desired final state.

Now we will exploit the (near) differential flatness of the PVTOL system to compute an optimal trajectory. The `pvtol` object is a `control.flatsys.FlatSystem` object. The forward map of x and u and their time derivatives to a flat output $\overline{z}$ is defined in the `pvtol` module as `_pvtol_flat_forward()`. The inverse map from $\overline{z}$ to x and u and its derivatives is defined as `_pvtol_flat_reverse()`. A `FlatSystem` object can be passed to the `control.flatsys.point_to_point()` function to compute a trajectory that starts at one point and ends at another and satisfies dynamics constraints (this is the primary advantage to finding a flat output for a system). Consider the following:

```
solve_time_start = time.time()
flat_traj = fs.point_to_point(
    pvtol, timepts=t2, x0=x0, u0=ueq, xf=xf, uf=ueq
)
solve_time_end = time.time()
print(f"Flat solve time: {solve_time_end - solve_time_start:.2f} s")
```

```
Flat solve time: 0.00 s
```

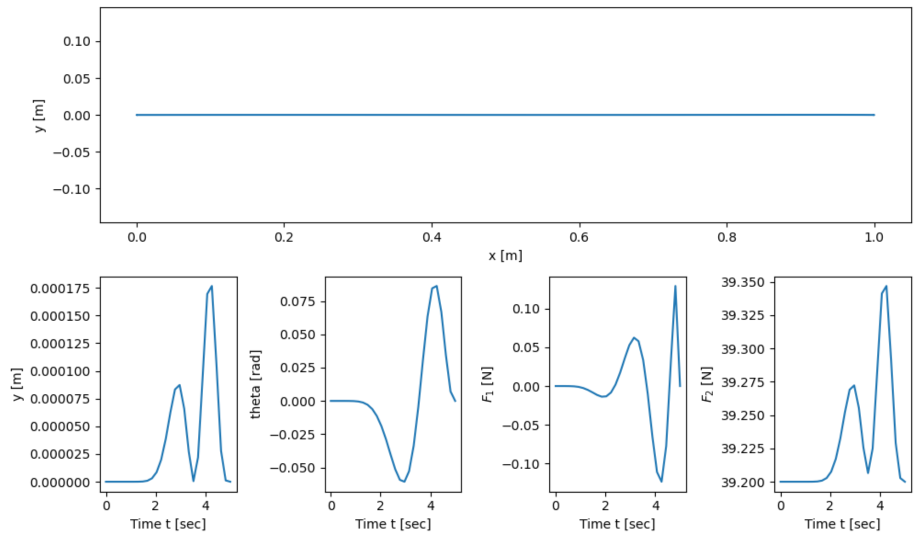
```
/Users/ricopicone/homepage/courses/me595-machine-learning-modern-control/_source/special-sol
  warn("System is only approximately flat (c != 0)")
```

Extract the optimal trajectory and plot the results as follows:

```
flat_traj.states, flat_traj.inputs = flat_traj.eval(t2)
print(f"Initial position: {flat_traj.states[0:2, 0]}")
print(f"Final position: {flat_traj.states[0:2, -1]}")
```

```
plot_results(t2, flat_traj.states, flat_traj.inputs)
plt.draw()
```

```
Initial position: [5.30661049e-14 0.00000000e+00]
Final position: [ 1.00000000e+00 -1.47098722e-11]
```



The flatness-based trajectory is smoother and faster to compute than the numerical optimization-based trajectories. The flatness-based trajectory ends essentially exactly at the desired final state.

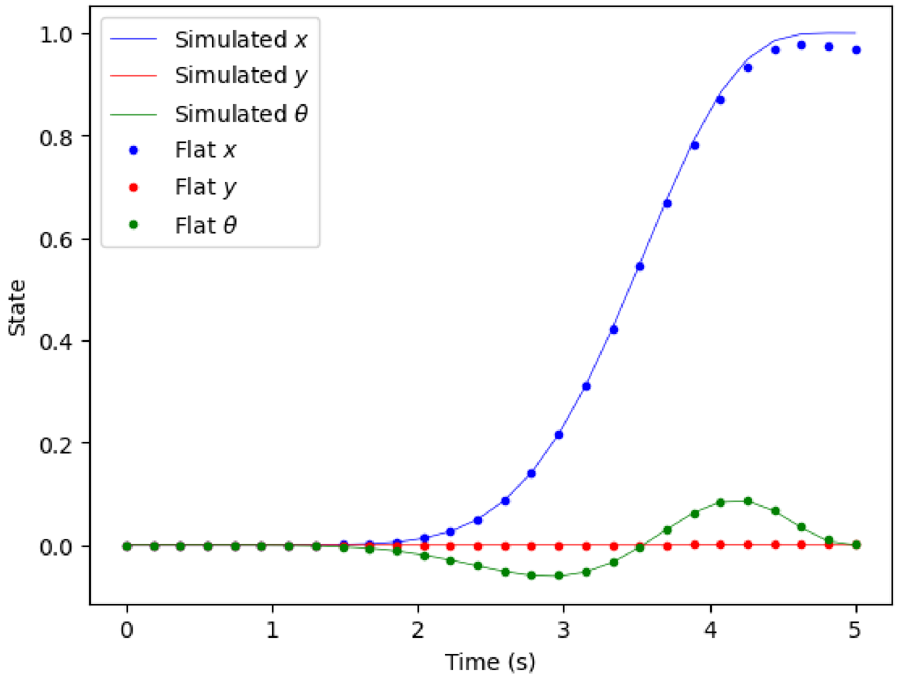
Note that the flatness-based trajectory is not necessarily optimal in the sense of minimizing a cost function. There is an option to add a cost function to the `point_to_point()` function, but I had trouble getting it to yield a good result.

Finally, let's attempt to use the flatness-based trajectory as the feedforward input to a state feedback controller.

```
_, y = control.forced_response(
    pvtol, T=t2, X0=x0, U=flat_traj.inputs, params={"c": 0}
)
```

Plot the position response of the feedforward system as follows:

```
fig, ax = plt.subplots()
ax.plot(t2, flat_traj.states[0], 'b', linewidth=0.5, label='Simulated $x$')
ax.plot(t2, flat_traj.states[1], 'r', linewidth=0.5, label='Simulated $y$')
ax.plot(t2, flat_traj.states[2], 'g', linewidth=0.5, label='Simulated $\\theta$')
ax.plot(t2, y[0], 'b.', label='Flat $x$')
ax.plot(t2, y[1], 'r.', label='Flat $y$')
ax.plot(t2, y[2], 'g.', label='Flat $\\theta$')
ax.set_xlabel('Time (s)')
ax.set_ylabel('State')
ax.legend()
plt.show()
```



The simulated position state and the predicted flat position state trajectories are quite close. With feedback control, the actual position state trajectories would likely be even closer to the predicted flat position state trajectories.