Differential Equations Primer

for SISO linear systems

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August 2018*

*Support provided by the Hal and Inge Marcus School of Engineering at Saint Martin's University (SMU).

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01 SISO linear systems

In engineering, we often consider the design, mathematical modeling, or analysis of machines, circuits, biological populations, etc. We call these, in aggregate, *systems*. The vast majority of systems we consider are *dynamic*: they change over time. We can analyze such systems by writing mathematical representations of appropriate physical laws.

systems
dynamic

01.01 Dynamic systems

For instance, a simple machine might have a link pinned and actuated by a motor at one end, as shown in Figure 01.1. The angle θ of the link might change with time, depending on the external forces acting on it, which include the motor torque. We could apply Newton's laws to describe this motion.

Assuming the link's weight creates a moment about the motor shaft much smaller than the torque T applied by the motor, and letting the link have mass moment of inertia I about the motor shaft, Newton's second law in its angular form yields

What type of mathematical object is this? The derivatives make it a *differential equation* with independent variable time t and dependent variables (functions of time) θ and T. The derivatives are all ordinary derivatives and not partial derivatives, so it is an *ordinary differential equation* (ODE). Let's assume the torque T applied by the motor is known

differentiality
ordinariness

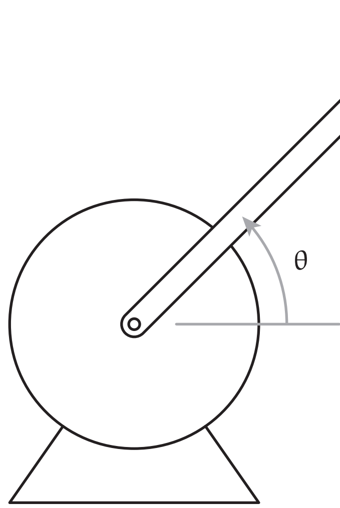

Figure 01.1: a simple machine consisting of a motor and a link.

and that the angle θ is unknown. The unknown dependent variable θ and its derivatives enter the differential equation *linearly*, making it a *linear ordinary differential equation*. **linearity**

Box 01.1 Course connections: Differential Equations, Computer Applications in Engineering

In Differential Equations, you spend a lot of time studying ODEs. This primer focuses on a specific subset of material from that course and presents one unified way to solve all such problems. This primer is, of course, no substitute for the course.

In Computer Applications in Engineering, you learned some fundamental numerical techniques for solving ODEs. These techniques apply directly to the ODEs presented in this primer, which focuses on *analytic* instead of *numerical* solutions.

dynamic systems **linear dynamic systems**

Dynamic systems that can be effectively described by such equations are called *linear dynamic systems*. Note that many are *approximately* linear. Therefore, we spend a great deal of time analyzing linear dynamic systems. In fact, early courses in physics and engineering—covering topics going by names such as *mechanics*, *electronics*, and *dynamics*—mostly consist of learning physical laws that have precisely this form. We have seen that, in at least one case (and, in fact, in many others), Newton's second law yields a linear system description. In electronics, one can effectively describe the voltage-current v-i relationships of discrete components such as capacitors and inductors with simple differential equations; letting a capacitor's capacitance be C and an inductor's inductance be L:

As we know, circuits often consist of several such components and can be described by combining these sorts of simple equations to form those that are more complex.

Box 01.2 Course connections: Physics I, Physics II, Dynamics

In Physics I and Dynamics, you were often applying Newton's second law to derive a system of equations. These equations were, in fact, ODEs!
In Physics II, you performed circuit analyses. Whenever a capacitor or an inductor were included in the circuit, the resulting equations were ODEs!

Box 01.3 Course connections: Mechatronics, System Dynamics and Control, Heat Transfer, Vibration Theory, etc.

In a great many of your Mechanical Engineering courses, you will encounter linear ODEs. Investing your time in this primer will pay dividends throughout. Note that more advanced solution techniques for multiple-input, multiple output (MIMO) systems, nonlinear systems, and distributed systems described by partial differential equations are beyond the scope of this primer. Where such systems arise in the ME curriculum, solution techniques will be discussed. However, throughout the curriculum, it is often assumed that you can solve linear ODEs without too much trouble.

01.02 Inputs

These system descriptions in the form linear ODEs often include "dependent" variables (meaning they're dependent on time) that can be considered *independent of the system's dynamics*. They are therefore prescribed externally and "input" to the system, thereby getting their name: *inputs*. **inputs**

What is and what is not an input depend on the system definition. For instance, in our motor-link example, above, we made the nebulous statement that the motor torque T was "known" and the angle θ was "unknown." Stated a bit more precisely, T was taken to be an *input*, whereas θ was not. This means the motor itself was *not* part of the system described by the ODE. However, we could have included it in the system. This would mean that T is *internal* to the system, which would require the application of additional physical laws to describe its electronic circuitry. The choice between these two options (and among others) depends on our design and analytical needs.

single input

A great number of systems of engineering interest have a *single input*. In our motor-link example, our single input was the torque T. In many electronic systems, a single voltage source supplies external power, and so is taken to be the system's single input. Even systems of great complexity can often be described as single input systems.

When a system has a single input, we will often use u to denote this variable.

01.03 Outputs

When designing and analyzing a system, certain dependent variables will be of particular interest. We call such variables *outputs*.

outputs

Often, only a single variable is of interest. In such cases, we say we have a *single output* system and we often denote the output with the symbol y. For instance, perhaps in our motor-link example we are interested in the angle θ, which we would then call an output.

single output

It turns out we're ignoring another class of variable[1] that we'll learn more about in Mechatronics. For now, let's assume that we're interested in every dependent variable, other than inputs, in our system.

An objection might be raised, here: how can it be that a single output y can describe the output of many systems if we're also going to take every dependent variable as an output? Won't more variables be required to describe the dynamics? For instance, if, in the motor-link example, we take the system to include the motor, we'll probably need the voltage and current therein to describe it. Together with θ, that's *three* outputs!

It turns out this can be assuaged by algebraic relationships among the variables. For instance, the current through the motor windings is proportional to the torque. Through such relationships, we can have our

[1] Variables of this class are called *state variables*.

cake (our single output) and eat most of it (eliminate extra dependent variables), too. The catch is that the *order* (highest derivative) of the differential equation describing a system typically increases through this process of variable elimination.

order

Box 01.4 Course connections: Mechatronics and System Dynamics and Control

We will learn in Mechatronics that we can express a system's dynamics as $n \in \mathbb{N}$ coupled first-order equations or as a single nth-order equation. In System Dynamics and Control, we'll learn to solve the coupled equations. In this primer, we'll learn to solve the single nth-order equation.

01.04 SISO linear systems

SISO systems

The result of all this is that we frequently encounter *single-input, single-output* (SISO) linear systems. The input-output dynamics of all these systems can be described by the linear ODE with constant coefficients[2] $a_i, b_j \in \mathbb{R}$ (which include such system parameters as mass and spring constants, capacitances and resistances, etc.), order n, and $m \leqslant n$ for $n, m \in \mathbb{N}_0$—as:

$$\begin{aligned} &\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = \\ b_m &\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \end{aligned} \tag{01.1}$$

The rest of this primer will take as its primary goal the description of a solution technique for Equation 01.1. *Solutions* will be functions $y(t)$ that satisfy Equation 01.1 in terms of parameters a_i, b_i and input $u(t)$, only.

solutions

[2]The restriction of the coefficients to temporal constants means we can add the qualifier "time-invariant" to such systems, which we will do in Mechatronics.

02

A unique solution exists

We're not yet sure if a solution even *exists* for Equation 01.1, and if it does, if it is *unique*—meaning it's the only solution.

existence

uniqueness

02.01 Existence and uniqueness

Rather than proving the existence and uniqueness of a solution, we will simply consider a theorem that states conditions under which existence and uniqueness do hold. In other words: *a unique solution exists*, and we'll explore the conditions for which this is true.

Let the *forcing function* f be the "right-hand side" of Equation 01.1:

forcing function

$$f(t) \equiv b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \tag{02.1}$$

Theorem 02.1 (existence and uniqueness)**.** A solution $y(t)$ of Equation 01.1 *exists* and is *unique* for $t \geqslant t_0$ if and only if both the following are specified:

1. n initial conditions

$$y(t_0),\ \frac{dy}{dt}\bigg|_{t=t_0}, \cdots, \frac{d^{n-1}y}{dt^{n-1}}\bigg|_{t=t_0} \quad \text{and}$$

2. a continuous forcing function $f(t)$ for $t \geqslant t_0$.

Assuming this theorem can be proved, and it can (Finan, 2018), we need only the initial conditions and the forcing function to guarantee ourselves there is a unique solution. Let's think about what this means in terms of the dynamics of a system. In a sense, if we know its initial state and the input or forcing[1] how it will behave for the rest of time is determined. When I say "in a sense," I mean that insofar as the system is well-described by Equation 01.1. The determinist implications of this must be understood to be approximate and limited in scope. I don't want to be responsible for creating a bunch of determinists (Hoefer, 2016).

Note however, that, given initial conditions and forcing, we only know *that* a unique solution exists, not *what* that solution is or *how* to find it.

02.02 Outlining a solution technique

It turns out that, given a forcing function and no initial conditions, several potential solutions can satisfy the ODE Equation 01.1; conversely, given certain initial conditions and no forcing function, several potential solutions satisfy the ODE. It is only when both initial conditions and a forcing function are given that a unique solution exists. It can be shown (Kreyszig, 2010) that the *general solution* y_g (also called the *total solution*)—actually a "family" of solutions with unknown constants—to Equation 01.1 is equal to the sum of two solutions that are often relatively easy to obtain:

general solution y_g

1. the *homogeneous solution* y_h, another family of solutions, this time to Equation 01.1 with $f(t) = 0$, and
2. the *particular solution* y_p, which satisfies Equation 01.1 sans initial conditions.

homogeneous solution y_h

particular solution y_p

That is,

$$y_g(t) = y_h(t) + y_p(t). \tag{02.2}$$

Methods for deriving homogeneous and particular solutions are the topics of Lecture 03 and Lecture 04.

The general solution y_g is still a *family* of solutions that all satisfy Equation 01.1 for a given forcing function f. It only becomes the *unique* solution, which we call the *specific solution* and typically denote simply y or (occasionally) y_s, once the initial conditions are applied to y_g.

specific solution

The diagram of Figure 02.1 illustrates this solution technique, with each arrow signifying that the block at its tail is supplied to and precedes the block at its head. Lectures proceed with the diagram:

[1]Usually, if we know the input u, it is trivial to apply Equation 02.1 to find the forcing function f. However, note that u must be differentiable m times.

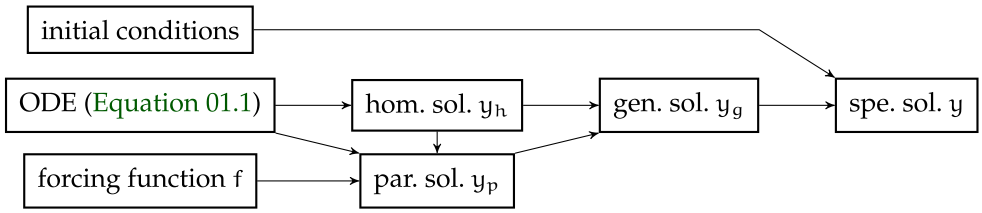

Figure 02.1: a diagram of the solution technique. Each arrow signifies that the block at its tail is supplied to and precedes the block at its head. The first column includes everything required to obtain a unique solution. The homogeneous y_h and particular y_p solutions of the second column sum to the general solution y_g. Applying the initial conditions to this yields the specific solution y.

- Lecture 03 describes how to obtain the homogeneous solution y_h from Equation 01.1 with the forcing function $f(t) = 0$;
- Lecture 04 describes how to derive the particular solution y_p from Equation 01.1 without the initial conditions for common forcing functions by a method called *undetermined coefficiencts*;
- Lecture 05 blows your mind by summing the homogeneous and particular solutions to obtain the general solution y_g; lest we be accused of dereliction of our duty to appear smarter than Business majors, this lecture also applies the initial conditions to the general solution to find constants introduced in the homogeneous solution to fully solve the differential equation—i.e., to obtain the specific solution y.

Box 02.1 Course connection: Differential Equations

The technique outlined here is probably quite similar to one described in your Differential Equations course. Terminology and notation may be different, so it may be worth correlating this primer with your previous coursework and text.

03

Homogeneous solution

The *homogeneous solution* (also called the *complementary solution*) to Equation 01.1 is the family of solutions[1] that satisfy Equation 01.1 with the forcing function $f(t) = 0$, but is not restricted by the initial conditions. The equation is

homogeneous solution

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = 0. \tag{03.1}$$

What function might solve this, the *homogeneous equation*? The natural exponential with base e is a good candidate, since it *is* its own derivative. It turns out that a *linear combination* (weighted sum) of *linearly independent* exponentials[2] is the family of solutions we're looking for.

homogeneous equation
linear combination
linearly independent

03.01 Characteristic equation and its roots

It can be shown that, for an exponential function $Ce^{\lambda t}$ with complex C and λ, the latter must satisfy

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0, \tag{03.2}$$

called the *characteristic equation*. It has n solutions or *roots* λ_i. When these roots are all *distinct*—meaning none is equal to another—the *homogeneous*

characteristic equation
roots
distinct
homogeneous solution

[1]This family of solutions is actually the *general* solution of the homogeneous equation, Equation 03.1. Lest it be confused with the general solution of Equation 01.1, we avoid calling it this in the following.

[2]The special case of repeated characteristic equation roots requires a slight modification of this statement, as we'll see. In this case, some of the exponentials pick up extra factors.

solution y_h to Equation 01.1 is

$$y_h(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t}. \tag{03.3}$$

03.02 Repeated roots

If a root is not distinct, it is said to have *multiplicity* μ equal to the number of its instances. So a root that appears thrice has multiplicity three. This multiplicity causes the linear combination of exponentials to be *degenerate* or *linearly dependent*. This is easily remedied, however, by augmenting the solution of Equation 03.3 with a polynomial term in t, as follows. Let there be n' distinct roots, each with multiplicity μ_i. Then the solution is:[3]

multiplicity

degenerate

linearly dependent

$$y_h(t) = \sum_{i=1}^{n'} \sum_{k=1}^{\mu_i} C_{ik} t^{(k-1)} e^{\lambda_{ik} t}. \tag{03.4}$$

Note that, as we would expect, when all roots are distinct, the factor t in each term has exponent zero, $t^0 = 1$, and we recover Equation 03.3.

03.03 What have we done?

We have found the homogeneous solution to Equation 01.1, which we know sums with another term to form its general solution. We should probably consider what this homogeneous solution looks like. It's a sum of weighted *complex* exponential functions of time. Complex solutions to the characteristic equation, Equation 03.2, always arise in complex conjugate pairs $\sigma \pm j\omega$, yielding terms like

This last identity is from a form of *Euler's formula*. The result shows us the possible forms of terms in the homogeneous solution. For a real root, $\omega = 0$ and a real exponential results (note, also, that the 2 goes away).

Euler's formula

[3] We use a double subscript ik, here, meaning the ith distinct root and the kth copy. Don't be alarmed: it's just to make sure there are enough distinct constants. If one prefers, the constants can be numbered 1 through n, but it's difficult to write that in summation form.

For imaginary roots, $\sigma = 0$ and a sinusoid results. For complex roots, a sinusoidal oscillation with an exponential envelope occurs.

Everything we know about the exponential also applies. For instance, if $\sigma < 0$, an exponential *decay* results, whereas if $\sigma > 0$, we get an exponential *growth*.

Example 03.03-1 A homogeneous solution

Find the homogeneous solution for the equation

$$\frac{d^5y}{dt^5} + 14\frac{d^4y}{dt^4} + 81\frac{d^3y}{dt^3} + 248\frac{d^2y}{dt^2} + 408\frac{dy}{dt} + 288y = f(t).$$

03.04 Exercises

See Appendix A for answers to the following exercises.

In all the following exercises, find the homogeneous solution y_p for Equation 01.1 with the order n and coefficients a_i given.

1. $n = 2$, $a_1 = -1$, $a_0 = -2$
2. $n = 2$, $a_1 = 6$, $a_0 = 9$
3. $n = 2$, $a_1 = 10$, $a_0 = 34$
4. $n = 5$, $a_4 = -7$, $a_3 = 32$, $a_2 = -124$, $a_1 = 256$, $a_0 = -192$

04

Particular solution

What effect does the forcing function f have on the solution? How might we solve for this effect, called the *particular solution*? One answer to the latter question will be given in this lecture: using the *method of undetermined coefficients*. Before we turn to this method, please recognize that other methods, such as the *method of variation* or *Laplace transforms* apply to more general forms of forcing f (although the integrals that accompany each may be unknown). When choosing the method of undetermined coefficients, we limit the scope of applicability to systems subjected to forcing functions that are complex exponentials (which include sinusoids) or polynomials. The principle of *superposition*, discussed in the Mechatronics course, allows us to construct solutions for linear systems subject to linear combinations of complex exponentials and polynomials.

particular solution
method of undetermined coefficients

04.01 Method of undetermined coefficients

The method is:

1. based on the form of the forcing function, *propose* an appropriate solution that includes undetermined coefficients (being careful to propose a solution linearly independent of the homogeneous solution),
2. substitute this proposed solution into the ODE, and
3. determine the undetermined coefficients by solving the algebraic system of equations that results from equating terms on each side of the equation.

propose

If there is, in fact, a solution to the algebraic system—that is, for the undetermined coefficients—*our proposed solution is our particular solution*,

$f(t)$	proposed $y_p(t)$	test value
k	K_1	0
kt^n	$K_n t^n + K_{n-1} t^{n-1} + \ldots + K_1 t + K_0$	0
$ke^{\lambda t}$	$K_1 e^{\lambda t}$	λ
$ke^{j\omega t}$	$K_1 e^{j\omega t}$	$j\omega$
$k\cos(\omega t + \phi)$	$K_1 \cos(\omega t) + K_2 \sin(\omega t)$	$j\omega$
$k\sin(\omega t + \phi)$	$K_1 \cos(\omega t) + K_2 \sin(\omega t)$	$j\omega$

Table 04.1: suggested particular solutions $y_p(t)$ (with undetermined coefficients) to propose for various forcing functions f. Let k, λ, ω, and ϕ be real constants and n be a positive integer. Furthermore, let K_i be the *undetermined coefficients*.

with coefficients now determined. However, if there is no solution,[1] our proposed solution is *not* our particular solution.

04.02 Some suggested solution proposals

How can one propose a solution? There are no clear answers other than "be clever or use known solutions." As remarkably unsatisfying as this is, we can still rejoice in being let off the hook, since we are certainly not clever. As mentioned, above, this method only really works if the forcing function is a complex exponential or a polynomial (but this can be extended, using superposition, to a large class of problems of interest). Table 04.1 is provided as a guide, but it essentially boils down to: if f is a complex exponential, propose that y_p is a complex exponential; if f is a polynomial, propose that y_p is a polynomial.

04.03 The parenthetical caveat

The only caveat, here, is the parenthetical warning from the three-step method about choosing a linearly independent solution. This is a result of a theorem we have not considered, here, but suffice it to say that, in order for our *general* solution to simply be the sum of the homogeneous and particular solutions, as we will propose in the next lecture, these two

[1] One should not simply throw up one's hands at a certain point and declare "there's no solution!" Rather, one should prove that there is none.

must be linearly independent. We will not only skip the details of why this is the case, but also the details of how to deal with it, opting instead for a simple recipe. The "test values" in Table 04.1 are to test whether or not the particular solution is a component of the homogeneous solution. If the test value is equal to any root of the characteristic equation of multiplicity μ, then the proposed solution should be multiplied by t^μ.

Example 04.03-1 A particular solution

Find the particular solution for the equation

$$\frac{d^5y}{dt^5} + 14\frac{d^4y}{dt^4} + 81\frac{d^3y}{dt^3} + 248\frac{d^2y}{dt^2} + 408\frac{dy}{dt} + 288y = f(t),$$

which is the same as that of Example 03.03-1, with

$$f(t) = a\cos(\omega t),$$

with $a \in \mathbb{R}$ and $\omega = 5$ rad/s.

04.04 Exercises

See Appendix A for answers to the following exercises.

In all the following exercises, find the particular solution y_p for Equation 01.1 with the order n, coefficients a_i, and forcing function f given.

1. $n = 2$, $a_1 = -1$, $a_0 = -2$, $f(t) = 3$
2. $n = 2$, $a_1 = 6$, $a_0 = 9$, $f(t) = 5e^{-3t}$
3. $n = 1$, $a_0 = 2$, $f(t) = 2\cos(3t)$
4. $n = 3$, $a_2 = 5$, $a_1 = 16$, $a_0 = 80$, $f(t) = t + 2$

05

General and specific solutions

We posited in Lecture 02 that the *general solution* to the ODE Equation 01.1 is

general solution

$$y_g(t) = y_h(t) + y_p(t). \tag{05.1}$$

We have not and will not prove this, but simply propose it to be the case. Working through a proof of this from, for instance, your differential equations textbook is of some value.

So, we already have y_h and y_p, so finding y_g is trivial. What type of object is y_g? The particular solution contributes only determined coefficients, but the homogeneous solution contributes n "unknown" constants C_i. This means y_g inherits those constants and therefore is a *family* of solutions.

This leads us to our final step: applying the initial conditions to find the specific constants C_i and thereby our *specific solution* y.

specific solution

"Applying" the initial conditions is simply to subject y_g to each of them. For instance, if we have two initial conditions, such as

we construct two algebraic equations

which is a system of algebraic equations from which the two unknown constants C_1 and C_2 (from the homogeneous solution) can be solved.

Example 05.00-1 A general and a specific solution

Find the general solution for the equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = f(t),$$

with

$$f(t) = a\cos(\omega t),$$

where $a \in \mathbb{R}$ and $\omega = 5$ rad/s. Apply the following initial conditions to obtain a specific solution:

$$y(0) = 3 \quad \text{and} \left.\frac{dy}{dt}\right|_{t=0} = 0.$$

05.01 Exercises

See Appendix A for answers to the following exercises.

In all the following exercises, find the specific solution y for Equation 01.1 with the order n, coefficients a_i, forcing function f, and initial conditions given. Note that the homogeneous and particular solutions from Lecture 04 apply to these problems, so they need not be re-derived.

1. $n = 2$, $a_1 = -1$, $a_0 = -2$, $f(t) = 3$, $y(0) = 2$, $dy/dt|_{t=0} = 0$
2. $n = 2$, $a_1 = 6$, $a_0 = 9$, $f(t) = 5e^{-3t}$, $y(0) = 0$, $dy/dt|_{t=0} = 0$
3. $n = 1$, $a_0 = 2$, $f(t) = 2\cos(3t)$, $y(0) = 4$
4. $n = 3$, $a_2 = 5$, $a_1 = 16$, $a_0 = 80$, $f(t) = t + 2$, $y(0) = 0$, $dy/dt|_{t=0} = 1$, $d^2y/dt^2|_{t=0} = 0$

A

Answers to exercises

A.01 Answers to the exercises of Lecture 03

Note: the indices of the constants are arbitrary.

1. $y_h(t) = C_1 e^{2t} + C_2 e^{-t}$.
2. $y_h(t) = C_1 e^{-3t} + C_2 t e^{-3t}$.
3. $y_h(t) = C_1 e^{(-5+j3)t} + C_2 e^{(-5-j3)t}$.
4. $y_h(t) = C_1 e^{3t} + C_2 e^{2t} + C_3 t e^{2t} + C_4 e^{j4t} + C_5 e^{-j4t}$.

A.02 Answers to the exercises of Lecture 04

1. $y_p(t) = -\frac{3}{2}$.
2. $y_p(t) = \frac{5}{2} t^2 e^{-3t}$.
3. $y_p(t) = \frac{4}{13}\cos(3t) + \frac{6}{13}\sin(3t)$.
4. $y_p(t) = \frac{1}{80} t + \frac{9}{400}$.

A.03 Answers to the exercises of Lecture 05

1. $y(t) = -\frac{3}{2} + \frac{7}{3} e^{-t} + \frac{7}{6} e^{2t}$.
2. $y(t) = \frac{5}{2} t^2 e^{-3t}$.
3. $y(t) = \frac{48}{13} e^{-2t} + \frac{4}{13}\cos(3t) + \frac{6}{13}\sin(3t)$.
4. $y(t) = -\frac{9}{1025} e^{-5t} - \frac{9}{656}\cos(4t) + \frac{619}{2624}\sin(4t) + \frac{1}{80} t + \frac{9}{400}$.

B

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