

## SISO linear systems

In engineering, we often consider the design, mathematical modeling, or analysis of machines, circuits, biological populations, etc. We call these, in aggregate, *systems*. The vast majority of systems we consider are *dynamic*: they change over time. We can analyze such systems by writing mathematical representations of appropriate physical laws.

systems  
dynamic

### 01.01 Dynamic systems

For instance, a simple machine might have a link pinned and actuated by a motor at one end, as shown in [Figure 01.1](#). The angle  $\theta$  of the link might change with time, depending on the external forces acting on it, which include the motor torque. We could apply Newton's laws to describe this motion.

Assuming the link's weight creates a moment about the motor shaft much smaller than the torque  $T$  applied by the motor, and letting the link have mass moment of inertia  $I$  about the motor shaft, Newton's second law in its angular form yields

What type of mathematical object is this? The derivatives make it a *differential equation* with independent variable time  $t$  and dependent variables (functions of time)  $\theta$  and  $T$ . The derivatives are all ordinary derivatives and not partial derivatives, so it is an *ordinary differential equation* (ODE). Let's assume the torque  $T$  applied by the motor is known

differentiality  
ordinariness

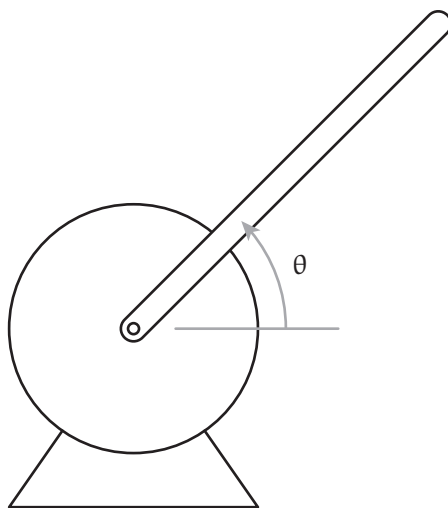


Figure 01.1: a simple machine consisting of a motor and a link.

and that the angle  $\theta$  is unknown. The unknown dependent variable  $\theta$  and its derivatives enter the differential equation *linearly*, making it a *linear ordinary differential equation*.

linearity

**Box 01.1 Course connections: Differential Equations, Computer Applications in Engineering**

In Differential Equations, you spend a lot of time studying ODEs. This primer focuses on a specific subset of material from that course and presents one unified way to solve all such problems. This primer is, of course, no substitute for the course.

In Computer Applications in Engineering, you learned some fundamental numerical techniques for solving ODEs. These techniques apply directly to the ODEs presented in this primer, which focuses on *analytic* instead of *numerical* solutions.

dynamic systems

linear dynamic systems

*Dynamic systems* that can be effectively described by such equations are called *linear dynamic systems*. Note that many are *approximately* linear. Therefore, we spend a great deal of time analyzing linear dynamic systems. In fact, early courses in physics and engineering—covering topics going by names such as *mechanics*, *electronics*, and *dynamics*—mostly consist of learning physical laws that have precisely this form. We have seen that, in at least one case (and, in fact, in many others), Newton's second law yields a linear system description. In electronics, one can effectively

describe the voltage-current  $v$ - $i$  relationships of discrete components such as capacitors and inductors with simple differential equations; letting a capacitor's capacitance be  $C$  and an inductor's inductance be  $L$ :

As we know, circuits often consist of several such components and can be described by combining these sorts of simple equations to form those that are more complex.

#### Box 01.2 Course connections: Physics I, Physics II, Dynamics

In Physics I and Dynamics, you were often applying Newton's second law to derive a system of equations. These equations were, in fact, ODEs!

In Physics II, you performed circuit analyses. Whenever a capacitor or an inductor were included in the circuit, the resulting equations were ODEs!

#### Box 01.3 Course connections: Mechatronics, System Dynamics and Control, Heat Transfer, Vibration Theory, etc.

In a great many of your Mechanical Engineering courses, you will encounter linear ODEs. Investing your time in this primer will pay dividends throughout. Note that more advanced solution techniques for multiple-input, multiple output (MIMO) systems, nonlinear systems, and distributed systems described by partial differential equations are beyond the scope of this primer. Where such systems arise in the ME curriculum, solution techniques will be discussed. However, throughout the curriculum, it is often assumed that you can solve linear ODEs without too much trouble.

## 01.02 Inputs

These system descriptions in the form linear ODEs often include "dependent" variables (meaning they're dependent on time) that can be considered *independent of the system's dynamics*. They are therefore prescribed externally and "input" to the system, thereby getting their name: *inputs*.

inputs

What is and what is not an input depend on the system definition. For instance, in our motor-link example, above, we made the nebulous statement that the motor torque  $T$  was “known” and the angle  $\theta$  was “unknown.” Stated a bit more precisely,  $T$  was taken to be an *input*, whereas  $\theta$  was not. This means the motor itself was *not* part of the system described by the ODE. However, we could have included it in the system. This would mean that  $T$  is *internal* to the system, which would require the application of additional physical laws to describe its electronic circuitry. The choice between these two options (and among others) depends on our design and analytical needs.

A great number of systems of engineering interest have a *single input*. In our motor-link example, our single input was the torque  $T$ . In many electronic systems, a single voltage source supplies external power, and so is taken to be the system’s single input. Even systems of great complexity can often be described as single input systems.

single  
input

When a system has a single input, we will often use  $u$  to denote this variable.

### 01.03 Outputs

When designing and analyzing a system, certain dependent variables will be of particular interest. We call such variables *outputs*.

outputs

Often, only a single variable is of interest. In such cases, we say we have a *single output* system and we often denote the output with the symbol  $y$ . For instance, perhaps in our motor-link example we are interested in the angle  $\theta$ , which we would then call an output.

single  
output

It turns out we’re ignoring another class of variable<sup>1</sup> that we’ll learn more about in Mechatronics. For now, let’s assume that we’re interested in every dependent variable, other than inputs, in our system.

An objection might be raised, here: how can it be that a single output  $y$  can describe the output of many systems if we’re also going to take every dependent variable as an output? Won’t more variables be required to describe the dynamics? For instance, if, in the motor-link example, we take the system to include the motor, we’ll probably need the voltage and current therein to describe it. Together with  $\theta$ , that’s *three* outputs!

It turns out this can be assuaged by algebraic relationships among the variables. For instance, the current through the motor windings is proportional to the torque. Through such relationships, we can have our

<sup>1</sup>Variables of this class are called *state variables*.

cake (our single output) and eat most of it (eliminate extra dependent variables), too. The catch is that the *order* (highest derivative) of the differential equation describing a system typically increases through this process of variable elimination. order

#### Box 01.4 Course connections: Mechatronics and System Dynamics and Control

We will learn in Mechatronics that we can express a system's dynamics as  $n \in \mathbb{N}$  coupled first-order equations or as a single  $n$ th-order equation. In System Dynamics and Control, we'll learn to solve the coupled equations. In this primer, we'll learn to solve the single  $n$ th-order equation.

## 01.04 SISO linear systems

The result of all this is that we frequently encounter *single-input, single-output* (SISO) linear systems. The input-output dynamics of all these systems can be described by the linear ODE with constant coefficients<sup>2</sup>  $a_i, b_j \in \mathbb{R}$  (which include such system parameters as mass and spring constants, capacitances and resistances, etc.), order  $n$ , and  $m \leq n$  for  $n, m \in \mathbb{N}_0$ —as: SISO systems

$$\begin{aligned} \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = \\ b_m \frac{d^m u}{dt^m} + b_{m-1} \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1 \frac{du}{dt} + b_0 u. \end{aligned} \quad (01.1)$$

The rest of this primer will take as its primary goal the description of a solution technique for [Equation 01.1](#). *Solutions* will be functions  $y(t)$  that satisfy [Equation 01.1](#) in terms of parameters  $a_i, b_i$  and input  $u(t)$ , only. solutions

<sup>2</sup>The restriction of the coefficients to temporal constants means we can add the qualifier "time-invariant" to such systems, which we will do in Mechatronics.