03

Homogeneous solution

The *homogeneous solution* (also called the *complementary solution*) to Equation 01.1 is the family of solutions[1] that satisfy Equation 01.1 with the forcing function $f(t) = 0$, but is not restricted by the initial conditions. The equation is

homogeneous solution

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = 0. \tag{03.1}$$

What function might solve this, the *homogeneous equation*? The natural exponential with base e is a good candidate, since it *is* its own derivative. It turns out that a *linear combination* (weighted sum) of *linearly independent* exponentials[2] is the family of solutions we're looking for.

homogeneous equation

linear combination

linearly independent

03.01 Characteristic equation and its roots

It can be shown that, for an exponential function $Ce^{\lambda t}$ with complex C and λ, the latter must satisfy

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0, \tag{03.2}$$

called the *characteristic equation*. It has n solutions or *roots* λ_i. When these roots are all *distinct*—meaning none is equal to another—the *homogeneous*

characteristic equation

roots

distinct

homogeneous solution

[1]This family of solutions is actually the *general* solution of the homogeneous equation, Equation 03.1. Lest it be confused with the general solution of Equation 01.1, we avoid calling it this in the following.

[2]The special case of repeated characteristic equation roots requires a slight modification of this statement, as we'll see. In this case, some of the exponentials pick up extra factors.

solution y_h to Equation 01.1 is

$$y_h(t) = \sum_{i=1}^{n} C_i e^{\lambda_i t}. \tag{03.3}$$

03.02 Repeated roots

If a root is not distinct, it is said to have *multiplicity* μ equal to the number of its instances. So a root that appears thrice has multiplicity three. This multiplicity causes the linear combination of exponentials to be *degenerate* or *linearly dependent*. This is easily remedied, however, by augmenting the solution of Equation 03.3 with a polynomial term in t, as follows. Let there be n' distinct roots, each with multiplicity μ_i. Then the solution is:[3]

multiplicity

degenerate

linearly dependent

$$y_h(t) = \sum_{i=1}^{n'} \sum_{k=1}^{\mu_i} C_{ik} t^{(k-1)} e^{\lambda_{ik} t}. \tag{03.4}$$

Note that, as we would expect, when all roots are distinct, the factor t in each term has exponent zero, $t^0 = 1$, and we recover Equation 03.3.

03.03 What have we done?

We have found the homogeneous solution to Equation 01.1, which we know sums with another term to form its general solution. We should probably consider what this homogeneous solution looks like. It's a sum of weighted *complex* exponential functions of time. Complex solutions to the characteristic equation, Equation 03.2, always arise in complex conjugate pairs $\sigma \pm j\omega$, yielding terms like

This last identity is from a form of *Euler's formula*. The result shows us the possible forms of terms in the homogeneous solution. For a real root, $\omega = 0$ and a real exponential results (note, also, that the 2 goes away).

Euler's formula

[3] We use a double subscript ik, here, meaning the ith distinct root and the kth copy. Don't be alarmed: it's just to make sure there are enough distinct constants. If one prefers, the constants can be numbered 1 through n, but it's difficult to write that in summation form.

For imaginary roots, $\sigma = 0$ and a sinusoid results. For complex roots, a sinusoidal oscillation with an exponential envelope occurs.

Everything we know about the exponential also applies. For instance, if $\sigma < 0$, an exponential *decay* results, whereas if $\sigma > 0$, we get an exponential *growth*.

Example 03.03-1 A homogeneous solution

Find the homogeneous solution for the equation

$$\frac{d^5 y}{dt^5} + 14\frac{d^4 y}{dt^4} + 81\frac{d^3 y}{dt^3} + 248\frac{d^2 y}{dt^2} + 408\frac{dy}{dt} + 288y = f(t).$$

03.04 Exercises

See Appendix A for answers to the following exercises.

In all the following exercises, find the homogeneous solution y_p for Equation 01.1 with the order n and coefficients a_i given.

1. $n = 2$, $a_1 = -1$, $a_0 = -2$
2. $n = 2$, $a_1 = 6$, $a_0 = 9$
3. $n = 2$, $a_1 = 10$, $a_0 = 34$
4. $n = 5$, $a_4 = -7$, $a_3 = 32$, $a_2 = -124$, $a_1 = 256$, $a_0 = -192$