04

Particular solution

What effect does the forcing function f have on the solution? How might we solve for this effect, called the *particular solution*? One answer to the latter question will be given in this lecture: using the *method of undetermined coefficients*. Before we turn to this method, please recognize that other methods, such as the *method of variation* or *Laplace transforms* apply to more general forms of forcing f (although the integrals that accompany each may be unknown). When choosing the method of undetermined coefficients, we limit the scope of applicability to systems subjected to forcing functions that are complex exponentials (which include sinusoids) or polynomials. The principle of *superposition*, discussed in the Mechatronics course, allows us to construct solutions for linear systems subject to linear combinations of complex exponentials and polynomials.

particular solution
method of undetermined coefficients

04.01 Method of undetermined coefficients

The method is:

1. based on the form of the forcing function, *propose* an appropriate solution that includes undetermined coefficients (being careful to propose a solution linearly independent of the homogeneous solution),
2. substitute this proposed solution into the ODE, and
3. determine the undetermined coefficients by solving the algebraic system of equations that results from equating terms on each side of the equation.

propose

If there is, in fact, a solution to the algebraic system—that is, for the undetermined coefficients—*our proposed solution is our particular solution*,

$f(t)$	proposed $y_p(t)$	test value
k	K_1	0
kt^n	$K_n t^n + K_{n-1} t^{n-1} + \ldots + K_1 t + K_0$	0
$ke^{\lambda t}$	$K_1 e^{\lambda t}$	λ
$ke^{j\omega t}$	$K_1 e^{j\omega t}$	$j\omega$
$k\cos(\omega t + \phi)$	$K_1 \cos(\omega t) + K_2 \sin(\omega t)$	$j\omega$
$k\sin(\omega t + \phi)$	$K_1 \cos(\omega t) + K_2 \sin(\omega t)$	$j\omega$

Table 04.1: suggested particular solutions $y_p(t)$ (with undetermined coefficients) to propose for various forcing functions f. Let k, λ, ω, and ϕ be real constants and n be a positive integer. Furthermore, let K_i be the *undetermined coefficients*.

with coefficients now determined. However, if there is no solution,[1] our proposed solution is *not* our particular solution.

04.02 Some suggested solution proposals

How can one propose a solution? There are no clear answers other than "be clever or use known solutions." As remarkably unsatisfying as this is, we can still rejoice in being let off the hook, since we are certainly not clever. As mentioned, above, this method only really works if the forcing function is a complex exponential or a polynomial (but this can be extended, using superposition, to a large class of problems of interest). Table 04.1 is provided as a guide, but it essentially boils down to: if f is a complex exponential, propose that y_p is a complex exponential; if f is a polynomial, propose that y_p is a polynomial.

04.03 The parenthetical caveat

The only caveat, here, is the parenthetical warning from the three-step method about choosing a linearly independent solution. This is a result of a theorem we have not considered, here, but suffice it to say that, in order for our *general* solution to simply be the sum of the homogeneous and particular solutions, as we will propose in the next lecture, these two

[1] One should not simply throw up one's hands at a certain point and declare "there's no solution!" Rather, one should prove that there is none.

must be linearly independent. We will not only skip the details of why this is the case, but also the details of how to deal with it, opting instead for a simple recipe. The "test values" in Table 04.1 are to test whether or not the particular solution is a component of the homogeneous solution. If the test value is equal to any root of the characteristic equation of multiplicity μ, then the proposed solution should be multiplied by t^μ.

Example 04.03-1 A particular solution

Find the particular solution for the equation

$$\frac{d^5y}{dt^5} + 14\frac{d^4y}{dt^4} + 81\frac{d^3y}{dt^3} + 248\frac{d^2y}{dt^2} + 408\frac{dy}{dt} + 288y = f(t),$$

which is the same as that of Example 03.03-1, with

$$f(t) = a\cos(\omega t),$$

with $a \in \mathbb{R}$ and $\omega = 5$ rad/s.

04.04 Exercises

See Appendix A for answers to the following exercises.

In all the following exercises, find the particular solution y_p for Equation 01.1 with the order n, coefficients a_i, and forcing function f given.

1. $n = 2$, $a_1 = -1$, $a_0 = -2$, $f(t) = 3$
2. $n = 2$, $a_1 = 6$, $a_0 = 9$, $f(t) = 5e^{-3t}$
3. $n = 1$, $a_0 = 2$, $f(t) = 2\cos(3t)$
4. $n = 3$, $a_2 = 5$, $a_1 = 16$, $a_0 = 80$, $f(t) = t + 2$