03.2 ss.ssmodel State and output equations

1 The state $\mathbf{x}$, input $\mathbf{u}$, and output $\mathbf{y}$ vectors interact through two equations:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \tag{1a}$$

$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t) \tag{1b}$$

where $\mathbf{f}$ and $\mathbf{g}$ are vector-valued functions that depend on the system. Together, they comprise what is called a **state-space model** of a system. Let's not glide past these equations, which will be our dear friends for the rest of our analytic lives. The first equation (1a) is called the **state equation**. Given state and input vectors at a moment in time, it's function $\mathbf{f}$ describes, *how the state is changing* (i.e. $d\mathbf{x}/dt$). Clearly, the state equation is a vector differential equation, which is equivalent to a system of first-order differential equations.[1]

2 In accordance with the definition of a state-determined system from Lecture 03.1 ss.svar, given an initial condition $\mathbf{x}(t_0)$ and input $\mathbf{u}$, the state $\mathbf{x}$ is determined for all $t \geqslant t_0$. The state-space model is precisely the "mathematical model" described in the definition of a state-determined system. Determining the state requires the solution—analytic or numerical—of the vector differential equation.

3 The second equation (1b) is *algebraic*. It expresses how the output $\mathbf{y}$ can be constructed from the state $\mathbf{x}$ and input $\mathbf{u}$. This means we must first solve the state equation (1a). Since the output $\mathbf{y}$ is a vector of variables of interest, the output equation is constructed in two steps: (1) define the output variables and (2) write them in terms of the state variables x_i and input variables u_j.

4 Just because we know that, for a state-determined system, there exists a solution to Equation 1a, doesn't mean we know how to find it. In general, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^n$ and $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^m$ can be nonlinear functions.[2]

[1] We'll learn how to solve such systems both analytically and numerically in later chapters.
[2] Technically, since $\mathbf{x}$ and $\mathbf{u}$ are themselves functions, $\mathbf{f}$ and $\mathbf{g}$ are *functionals*.

We don't know how to solve most nonlinear state equations analytically. An additional complication can arise when, in addition to states and inputs, system parameters are themselves time-varying (note the explicit time t argument of f and g). Fortunately, often a linear model is sufficiently effective.[3]

5 A **linear, time-invariant** (LTI) system has state-space model

$$\frac{dx}{dt} = Ax + Bu \tag{2a}$$

$$y = Cx + Du \tag{2b}$$

where

- A is an matrix that describes how the changes the ,
- B is an matrix that describes how the changes the ,
- C is an matrix that describes how the contributes to the , and
- D is an matrix that describes how the contributes to the .

In the next two lectures, we will learn how to derive a state-space model—for linear systems, how to find A, B, C, and D—for a system *from its linear graph*. This is the link between the linear graph model and the state-space model.

[3] A later lecture will describe the process of deriving a "linearized" model from a nonlinear one.