03.7 ss.ss2tf2io Bridge between state-space and io differential equations

1 The **Laplace transform** $\mathcal{L}$ is cool af. It is used to solve differential equations and define the **transfer function** H: you know, just another awesome dynamic system representation. For now, we'll use it as a bridge between state-space and input/output differential equation representations, merely waving at transfer functions as we pass through. Later, transfer functions will be considered extensively.

Transfer functions

2 Let a system have an input u and an output y. Let the Laplace transform of each be denoted U and Y, both functions of complex Laplace transform variable s. A **transfer function** H is defined as the ratio of the Laplace transform of the output over the input:

$$H(s) = \frac{Y(s)}{U(s)}. \tag{1}$$

3 The transfer function is exceedingly useful in many types of analysis. One of its most powerful aspects is that it gives us access to thinking about systems as operating on an input u and yielding an output y.

Bridging transfer functions and io differential equations

4 Consider a dynamic system described by the *input-output differential equation*—with variable y representing the *output*, dependent variable time t, variable u representing the *input*, constant coefficients a_i, b_j, order n, and $m \leqslant n$ for $n \in \mathbb{N}_0$—as:

$$\begin{aligned}\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = \\ b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u.\end{aligned} \tag{2}$$

5 The **Laplace transform** $\mathcal{L}$ of Eq. 2 yields something interesting (assuming zero initial conditions):

$$\begin{aligned}
\mathcal{L}\left(\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y\right) &= \\
\mathcal{L}\left(b_m\frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u\right) &\Rightarrow \\
\mathcal{L}\left(\frac{d^n y}{dt^n}\right) + a_{n-1}\mathcal{L}\left(\frac{d^{n-1}y}{dt^{n-1}}\right) + \cdots + a_1\mathcal{L}\left(\frac{dy}{dt}\right) + a_0\mathcal{L}(y) &= \\
b_m\mathcal{L}\left(\frac{d^m u}{dt^m}\right) + b_{m-1}\mathcal{L}\left(\frac{d^{m-1}u}{dt^{m-1}}\right) + \cdots + b_1\mathcal{L}\left(\frac{du}{dt}\right) + b_0\mathcal{L}(u) &\Rightarrow \\
s^n Y + a_{n-1}s^{n-1}Y + \cdots + a_1 sY + a_0 Y &= \\
b_m s^m U + b_{m-1}s^{m-1}U + \cdots + b_1 sU + b_0 U.
\end{aligned}$$

Solving for Y,

The inverse Laplace transform $\mathcal{L}^{-1}$ of Y is the **forced response**. However, this is not our primary concern; rather, we are interested to solve for the transfer function H as the ratio of the output transform Y to the input transform U, i.e.

$$H(s) \equiv \frac{Y(s)}{U(s)} \tag{3}$$

$$= \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}. \tag{4}$$

6 Exactly the reverse procedure, then, can be used to convert a given transfer function to an input-output differential equation.

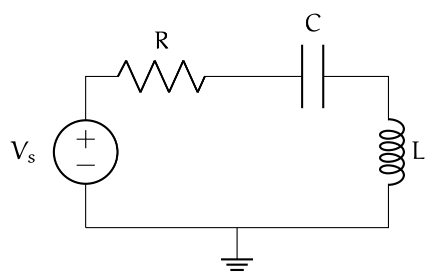
Example 03.7 ss.ss2tf2io-1

re: A circuit transfer function

The circuit shown has input-output differential equation

$$L\frac{d^2 v_L}{dt^2} + R\frac{dv_L}{dt} + \frac{1}{C}v_L = L\frac{d^2 V_s}{dt^2}.$$

What is the transfer function from V_s to v_L?



Bridging transfer functions and state-space models

7 Given a system in the standard form of a state equation,

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u},$$

we take the Laplace transform to yield, assuming zero initial conditions,

which can be solved for the state:

$$(5)$$

where I is the identity matrix with the same dimension as that of A. The standard form of the output equation yields the output solution

$$\mathbf{Y} = H\mathbf{U}, \tag{6}$$

where we define the **matrix transfer function** H to be

The element H_{ij} is a transfer function from the jth input U_j to the ith output Y_i.

8 The reverse procedure of deriving a state-space model from a transfer function is what is called a **state-space realization**, which is not a unique operation (there are different realizations for a single transfer function) and is not considered here.

Example 03.7 ss.ss2tf2io-2

re: Matrix transfer function from state-space

Given the linear state-space model

$$\dot{x} = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u,$$

derive the matrix transfer function.

Example 03.7 ss.ss2tf2io-3

re: state-space to io differential equations

For the following state-space model, derived in Example 03.4 ss.nt2ss-1, derive the io differential equations for each output variable:

$$\frac{dx}{dt} = \begin{bmatrix} \frac{-1}{R_1 C} & \frac{-1}{C} \\ 1/L & -R_2/L \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1 C} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \\ -1/R_1 & 0 \\ 0 & R_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/R_1 \\ 0 \end{bmatrix} u.$$

The output variables are i_L, I_S, and v_{R_2}.