05.2 lti.equistab   Equilibrium and stability properties

1   For a system with LTI state-space model $\dot{x} = Ax + Bu$, $y = Cx + Du$, the model is in an **equilibrium** state $\bar{x}$ if $\dot{x} = 0$. This implies $A\bar{x} + Bu = 0$. For constant input $\bar{u}$, this implies

If A is invertible,[1] as is often the case, there is a unique solution for a single **equilibrium** state:

Definition 05 lti.1: Stability

If x is perturbed from an equilibrium state $\bar{x}$, the response $x(t)$ can:

1. asymptotically return to $\bar{x}$
2. diverge from $\bar{x}$
3. remain perturbed or oscillate about $\bar{x}$ with a constant amplitude

2   A **phase portrait** is a parametric plot of state variable **trajectories**, with time implicit. Phase portraits are exceptionally useful for understanding nonlinear systems, but they also give us a nice way to understand stability, as in Figure equistab.1.

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[1] If A is not invertible, the system has at least one eigenvalue equal to zero, which yields an *equilibrium subspace* equal to an offset (by $B\bar{u}$) of the null space of the state space $\mathbb{R}^n$.
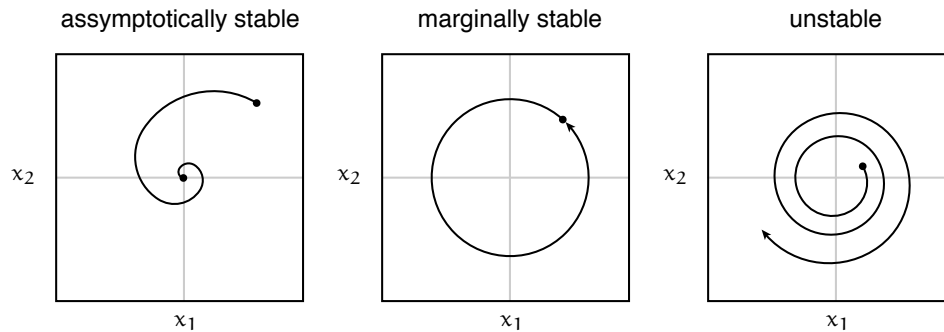
Figure equistab.1: a phase-portrait demonstration of (left) asymptotic stability, (center) marginal stability, and (right) instability for a second-order system.

3   These definitions of stability can be interpreted in terms of the free response of a system, as described, below.

Stability defined by the free response

4   Using the concept of the free response (no inputs, just initial conditions), we define the following types of stability for LTI systems[2].

1. An LTI system is **asymptotically stable** if the free response approaches an equilibrium state as time approaches infinity.
2. An LTI system is **unstable** if the free response grows without bound as time approaches infinity.
3. An LTI system is **marginally stable** if the free response neither decays nor grows but remains constant or oscillates as time approaches infinity.

5   These statements imply that the free response alone governs stability. Recall that the free response y_{fr} of a system with characteristic equation roots λ_i with multiplicity m_i, for constants C_i, is

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[2] Nise, 2015.

Each term will either decay to zero, remain constant, or increase without bound—depending on the sign of the *real part* of the corresponding root of the characteristic equation $\text{Re}(\lambda_i)$.

6 In other words, for an LTI system, the following statements hold.

1. An LTI system is *asymptotically stable* if, for all λ_i, $\text{Re}(\lambda_i) < 0$.
2. An LTI system is *unstable* if, for any λ_i, $\text{Re}(\lambda_i) > 0$.
3. An LTI system is *marginally stable* if,

    a) for all λ_i, $\text{Re}(\lambda_i) \leqslant 0$ *and*
    b) at least one $\text{Re}(\lambda_i) = 0$ *and*
    c) no λ_i for which $\text{Re}(\lambda_i) = 0$ has multiplicity $m_i > 1$.

Stability defined by the forced response

7 An alternate formulation of the stability definitions above is called the **bounded-input bounded-output** (BIBO) definition of stability, and states the following[3].

1. A system is **BIBO stable** if every bounded input yields a bounded output.
2. A system is **BIBO unstable** if any bounded input yields an unbounded output.

8 In terms of BIBO stability, **marginal stability**, then, means that a system has a bounded response to some inputs and an unbounded response to others. For instance, a second-order undamped system response to a sinusoidal input at the natural frequency is unbounded, whereas every other input yields a bounded output.

9 Although we focus on the definitions of stability in terms of the free response, it is good to understand BIBO stability, as well.

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[3] Nise, 2015.