05.2 lti.equistab Equilibrium and stability properties

1 For a system with LTI state-space model $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$, $\mathbf{y} = C\mathbf{x} + D\mathbf{u}$, the model is in an **equilibrium** state $\overline{\mathbf{x}}$ if $\dot{\mathbf{x}} = 0$. This implies $A\overline{\mathbf{x}} + B\mathbf{u} = 0$. For constant input $\overline{\mathbf{u}}$, this implies

If A is invertible,[1] as is often the case, there is a unique solution for a single **equilibrium** state:

Definition 05 lti.1: Stability

If $\mathbf{x}$ is perturbed from an equilibrium state $\overline{\mathbf{x}}$, the response $\mathbf{x}(t)$ can:

1. asymptotically return to $\overline{\mathbf{x}}$
2. diverge from $\overline{\mathbf{x}}$
3. remain perturbed or oscillate about $\overline{\mathbf{x}}$ with a constant amplitude

2 A **phase portrait** is a parametric plot of state variable **trajectories**, with time implicit. Phase portraits are exceptionally useful for understanding nonlinear systems, but they also give us a nice way to understand stability, as in Figure equistab.1.

[1] If A is not invertible, the system has at least one eigenvalue equal to zero, which yields an *equilibrium subspace* equal to an offset (by $B\overline{\mathbf{u}}$) of the null space of the state space $\mathbb{R}^n$.

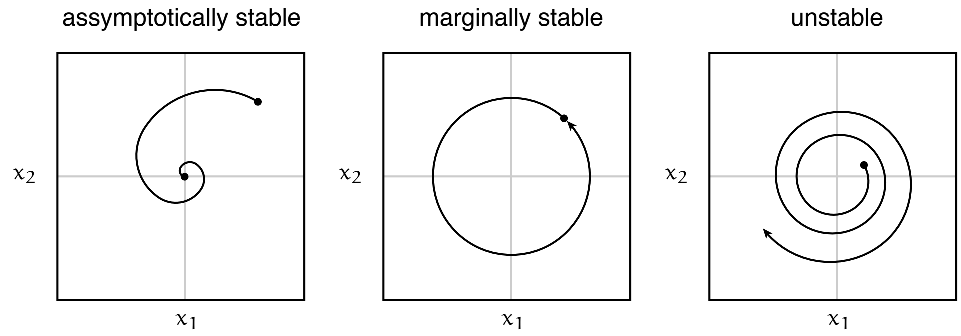

Figure equistab.1: a phase-portrait demonstration of (left) asymptotic stability, (center) marginal stability, and (right) instability for a second-order system.

3 These definitions of stability can be interpreted in terms of the free response of a system, as described, below.

Stability defined by the free response

4 Using the concept of the free response (no inputs, just initial conditions), we define the following types of stability for LTI systems[2].

1. An LTI system is **asymptotically stable** if the free response approaches an equilibrium state as time approaches infinity.
2. An LTI system is **unstable** if the free response grows without bound as time approaches infinity.
3. An LTI system is **marginally stable** if the free response neither decays nor grows but remains constant or oscillates as time approaches infinity.

5 These statements imply that the free response alone governs stability. Recall that the free response y_{fr} of a system with characteristic equation roots λ_i with multiplicity m_i, for constants C_i, is

[2] Nise, 2015.

Each term will either decay to zero, remain constant, or increase without bound—depending on the sign of the *real part* of the corresponding root of the characteristic equation $\mathrm{Re}(\lambda_i)$.

6 In other words, for an LTI system, the following statements hold.

1. An LTI system is *asymptotically stable* if, for all λ_i, $\mathrm{Re}(\lambda_i) < 0$.
2. An LTI system is *unstable* if, for any λ_i, $\mathrm{Re}(\lambda_i) > 0$.
3. An LTI system is *marginally stable* if,
    a) for all λ_i, $\mathrm{Re}(\lambda_i) \leqslant 0$ *and*
    b) at least one $\mathrm{Re}(\lambda_i) = 0$ *and*
    c) no λ_i for which $\mathrm{Re}(\lambda_i) = 0$ has multiplicity $m_i > 1$.

Stability defined by the forced response

7 An alternate formulation of the stability definitions above is called the **bounded-input bounded-output** (BIBO) definition of stability, and states the following[3].

1. A system is **BIBO stable** if every bounded input yields a bounded output.
2. A system is **BIBO unstable** if any bounded input yields an unbounded output.

8 In terms of BIBO stability, **marginal stability**, then, means that a system has a bounded response to some inputs and an unbounded response to others. For instance, a second-order undamped system response to a sinusoidal input at the natural frequency is unbounded, whereas every other input yields a bounded output.

9 Although we focus on the definitions of stability in terms of the free response, it is good to understand BIBO stability, as well.

[3] Nise, 2015.