

05.3 lti.vib Vibration isolation table analysis

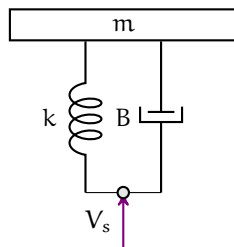


Figure vib.1: a vibration isolation table schematic with input velocity V_s .

- 1 In this example, we exercise many of the methods for modeling and analysis explored thus far.
- 2 Given the vibration isolation table model in [Figure vib.1](#)—with $m = 320$ kg, $k = 16000$ N/m, and $B = 1200$ N-m/s—derive:
 1. a linear graph model,
 2. a state-space model,
 3. the equilibrium state \bar{x} for the unit step input,
 4. a transfer function model,
 5. an input-output differential equation model with input V_s and output v_m ,
 6. a solution for $v_m(t)$ for a unit step input $V_s(t) = 1$ m/s for $t \geq 0$,
 7. the system's stability.

Linear graph and state-space models

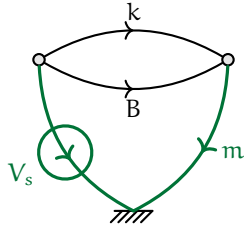


Figure vib.2: linear graph of the isolation table.

3 The linear graph and normal tree are shown in [Figure vib.2](#). Note that there is an equilibrium for this system, so we are justified in ignoring gravity and referencing any displacements to the static equilibrium position.⁴ The state variables are the velocity of the mass v_m and the force through the spring f_k and the system order is $n = 2$. The input, state, and output vectors are

$$\mathbf{u} = [V_s] \quad \mathbf{x} = \begin{bmatrix} v_m \\ f_k \end{bmatrix} \quad \mathbf{y} = [v_m].$$

The elemental equations are as follows.

$$\begin{array}{l|l} m & \dot{v}_m = \frac{1}{m} f_m \\ k & \dot{f}_k = k v_k \\ B & f_B = B v_B \end{array}$$

The continuity and compatibility equations are as follows.

branch	continuity equation	link	compatibility equation
m	$f_m = f_k + f_B$	k	$v_k = V_s - v_m$
		B	$v_B = V_s - v_m$

The state equation can be found by substituting the continuity and compatibility equations into the elemental equations, and eliminating f_B , to

⁴For a discussion of this ignoring of gravity, see [Lec. 05.4 lti.ghost](#).

yield

$$\dot{\mathbf{x}} = \begin{bmatrix} -B/m & 1/m \\ -k & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} B/m \\ k \end{bmatrix} \mathbf{u} \quad (1a)$$

$$\mathbf{y} = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \end{bmatrix} \mathbf{u}. \quad (1b)$$

Equilibrium

4 Let's check to see if A is invertible by trying to compute its inverse:

$$A^{-1} = \begin{bmatrix} -B/m & 1/m \\ -k & 0 \end{bmatrix}^{-1} \quad (2)$$

$$= \frac{1}{k/m} \begin{bmatrix} 0 & -1/m \\ k & -B/m \end{bmatrix} \quad (3)$$

So it has an inverse, after all! Let's use this to compute the equilibrium state:

$$\bar{\mathbf{x}} = -A^{-1} B \bar{\mathbf{u}} \quad (4)$$

$$= \frac{-m}{k} \begin{bmatrix} 0 & -1/m \\ k & -B/m \end{bmatrix} \begin{bmatrix} B/m \\ k \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \quad (5)$$

$$= \frac{-m}{k} \begin{bmatrix} -k/m \\ 0 \end{bmatrix} \quad (6)$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (7)$$

So the system is in equilibrium with $v_m = 1 \text{ m/s}$ and $f_k = 0 \text{ N}$. Since v_m is also our output, we expect 1 m/s to be our steady-state output value.

Transfer function model

5 The transfer function $H(s) = V_m(s)/V_s(s)$ will be used as a bridge to the input-output differential equation. The transfer function can be found from the usual formula, from [Lecture 03.7 ss.ss2tf2io](#),

$$H(s) = C(sI - A)^{-1} B + D. \quad (8)$$

Let's first compute $(sI - A)^{-1}$.⁵

$$(sI - A)^{-1} = \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -B/m & 1/m \\ -k & 0 \end{bmatrix} \right)^{-1} \quad (9a)$$

$$= \begin{bmatrix} s + B/m & -1/m \\ k & s \end{bmatrix}^{-1} \quad (9b)$$

$$= \frac{1}{(s + B/m)(s) - (-1/m)(k)} \begin{bmatrix} s & 1/m \\ -k & s + B/m \end{bmatrix} \quad (9c)$$

$$= \frac{1}{s^2 + (B/m)s + k/m} \begin{bmatrix} s & 1/m \\ -k & s + B/m \end{bmatrix} \quad (9d)$$

Now we're ready to compute the entirety of H :

$$H(s) = \frac{1}{s^2 + (B/m)s + k/m} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & 1/m \\ -k & s + B/m \end{bmatrix} \begin{bmatrix} B/m \\ k \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} \quad (10a)$$

$$= \frac{1}{s^2 + (B/m)s + k/m} \begin{bmatrix} s & 1/m \end{bmatrix} \begin{bmatrix} B/m \\ k \end{bmatrix} \quad (10b)$$

$$= \frac{(B/m)s + k/m}{s^2 + (B/m)s + k/m}. \quad (10c)$$

Input-output differential equation

6 The input-output differential equation can be found from the reverse of the procedure in [Lecture 03.7 ss.ss2tf2io](#). Beginning from the transfer function,

$$\frac{V_m}{V_s} = \frac{(B/m)s + k/m}{s^2 + (B/m)s + k/m} \Rightarrow \quad (11a)$$

$$(s^2 + (B/m)s + k/m) V_m = ((B/m)s + k/m) V_s \Rightarrow \quad (11b)$$

$$\ddot{v}_m + (B/m)\dot{v}_m + (k/m)v_m = (B/m)\dot{V}_s + (k/m)V_s. \quad (11c)$$

Step response

7 The step response is found using superposition and the derivative property of LTI systems. The forcing function $f(t) = (B/m)\dot{V}_s + (k/m)V_s$ is

⁵See (Rowell and Wormley, 1997, Sec. A.4.3) for details on the matrix inverse.

composed of two terms, one of which has a derivative of the input V_s . Rather than attempting to solve the entire problem at once, we choose to find the response for a forcing function $f(t) = 1$ (for $t \geq 0$)—that is, the unit step response—and use superposition and the derivative property of LTI systems to calculate the composite response.

Unit step response

8 The characteristic equation of Equation 11c is

$$\lambda^2 + (B/m)\lambda + k/m = 0 \Rightarrow \quad (12a)$$

$$= -\frac{B}{2m} \pm \frac{\sqrt{B^2 - 4mk}}{2m} \Rightarrow \quad (12b)$$

$$\lambda_{1,2} = -1.875 \pm j6.818. \quad (12c)$$

The roots are complex, so the system will have a damped sinusoidal step response. Let $\sigma = -1.875$ and $\omega = 6.818$ such that $\lambda_{1,2} = \sigma \pm j\omega$. The homogeneous solution is

$$v_{m_h}(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \quad (13)$$

In this form, C_1 and C_2 are complex. It is somewhat easier to deal with

$$v_{m_h}(t) = C_1 e^{\sigma t} e^{j\omega t} + C_2 e^{\sigma t} e^{-j\omega t} \quad (14a)$$

$$= e^{\sigma t} (C_1 \cos \omega t + jC_1 \sin \omega t + C_2 \cos \omega t - jC_2 \sin \omega t) \quad (14b)$$

$$= e^{\sigma t} ((C_1 + C_2) \cos \omega t + j(C_1 - C_2) \sin \omega t). \quad (14c)$$

Let $C_3 = C_1 + C_2$ and $C_4 = j(C_1 - C_2)$ such that

$$v_{m_h}(t) = e^{\sigma t} (C_3 \cos \omega t + C_4 \sin \omega t). \quad (15)$$

This is a decaying (because $\sigma < 0$) sinusoid. A nice aspect of this new form is that C_3 and C_4 are real.

9 Now, the particular solution can be found by assuming a solution of the form $v_{m_p}(t) = K$ for $t \geq 0$. Substituting this into Equation 11c (with forcing $f(t) = 1$, we attempt to find a solution for K (that is, *determine it*):

$$(k/m)K = 1 \Rightarrow K = m/k. \quad (16)$$

Therefore, $v_{m_p}(t) = m/k$ is a solution, and therefore the general solution is

$$v_{m_{\text{step}}}(t) = v_{m_h}(t) + v_{m_p}(t) \quad (17a)$$

$$= e^{\sigma t} (C_3 \cos \omega t + C_4 \sin \omega t) + m/k. \quad (17b)$$

This leaves the specific solution, to be found applying the initial conditions (assumed to be zero). Before we do so, however, we need the time-derivative of the $v_{m_{\text{step}}}$:

$$\dot{v}_{m_{\text{step}}}(t) = e^{\sigma t} ((C_3 \sigma + C_4 \omega) \cos(\omega t) + (C_4 \sigma - C_3 \omega) \sin(\omega t)). \quad (18)$$

Now, applying the initial conditions,

$$v_{m_{\text{step}}}(0) = 0 \Rightarrow \quad (19a)$$

$$C_3 = -m/k \quad (19b)$$

$$\dot{v}_{m_{\text{step}}}(0) = 0 \Rightarrow 0 = C_3 \sigma + C_4 \omega \Rightarrow \quad (19c)$$

$$C_4 = \frac{\sigma}{\omega} \cdot \frac{m}{k}. \quad (19d)$$

10 It's good form to re-write this as a single sinusoid:

$$v_{m_{\text{step}}}(t) = v_{m_h}(t) + v_{m_p}(t) \quad (20a)$$

$$= A_1 e^{\sigma t} \cos(\omega t + \psi_1) + m/k \quad (20b)$$

where we have used [Lecture 01.2 math.trig](#) to find

$$A_1 = \sqrt{C_3^2 + C_4^2} \quad (21a)$$

$$\psi_1 = -\arctan(C_4/C_3). \quad (21b)$$

Superposition and the derivative property

11 Recall that the actual forcing function is a linear combination of the input and its time-derivative. Therefore, it is expedient to re-write the time-derivative of the unit step response:

$$\dot{v}_{m_{\text{step}}}(t) = A_1 e^{\sigma t} (\sigma \cos(\omega t + \psi_1) - \omega \sin(\omega t + \psi_1)) \quad (22a)$$

$$= A_1 A_2 e^{\sigma t} \cos(\omega t + \psi_1 + \psi_2) \quad (22b)$$

where

$$A_2 = \sqrt{\sigma^2 + \omega^2} \quad (22c)$$

$$\psi_2 = -\arctan(-\omega/\sigma). \quad (22d)$$

Finally, applying superposition and the derivative rule of LTI systems,

$$v_m(t) = (B/m)\dot{v}_{m_{\text{step}}}(t) + (k/m)v_{m_{\text{step}}} \quad (23a)$$

$$= \frac{B}{m}A_1A_2e^{\sigma t}\cos(\omega t + \psi_1 + \psi_2) + \frac{k}{m}A_1e^{\sigma t}\cos(\omega t + \psi_1) + 1. \quad (23b)$$

This is the solution!

12 It's worth plotting the response. Begin by defining the system parameters.

```
m = 320; % kg ... mass
k = 16000; % N/m ... spring constant
B = 1200; % N-m/s ... damping coefficient
```

Now define the secondary parameters.

```
lambda = -B/(2*m)+[-1,1]*sqrt(B^2-4*m*k)/(2*m);
sigma = real(lambda(1));
omega = imag(lambda(2));
K = m/k;
C3 = -m/k;
C4 = sigma/omega*m/k;
A1 = sqrt(C3^2+C4^2);
psi1 = -atan2(C4,C3);
A2 = sqrt(sigma^2+omega^2);
psi2 = -atan2(-omega,sigma);
```

Finally, the solution for $v_m(t)$ can be defined as an anonymous function.

```
vm = @(t) ...
    A1*A2*B/m*exp(sigma*t).*cos(omega*t+psi1+psi2)+...
    A1*k/m*exp(sigma*t).*cos(omega*t+psi1)+...
    1;
```

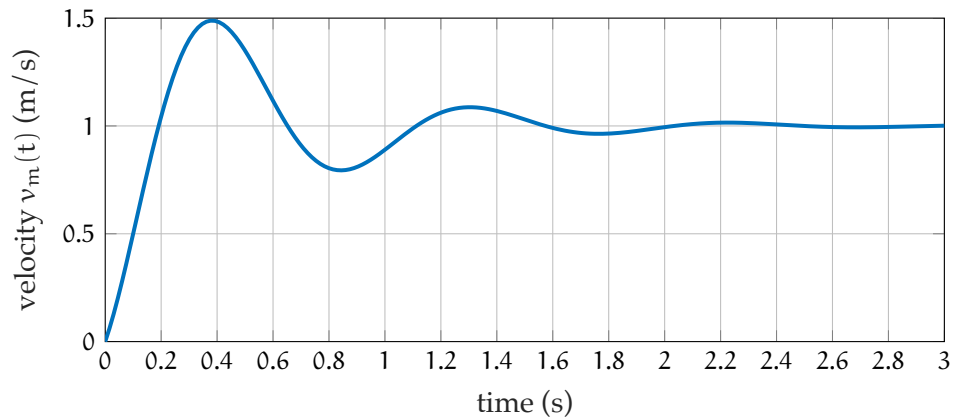


Figure vib.3: vibration table step response $v_m(t)$.

Now, plot over the first few seconds. The results are shown in [Figure vib.3](#).

```
t_a = linspace(0,3,200);
h = figure;
p = plot(t_a,vm(t_a),'linewidth',1.5);
xlabel('time (s)')
ylabel('velocity $v_m(t)$ (m/s)',...
       'interpreter','latex');
grid on
hgsave(h,'figures/temp');
```

13 Note that the steady-state output value agrees with that predicted by the equilibrium analysis, above.

Stability

14 We have learned what we need in order to analyze the system's stability. The roots of the characteristic equation were $\lambda_{1,2} = -1.875 \pm j6.818$, which clearly *all* have negative real parts, and therefore the system is *asymptotically stable*.