07.1 ssresp.response Solving for the state-space response

1 In this lecture, we solve the state equation for the **state response** $x(t)$ and substitute this into the output equation for the **output response** $y(t)$.

State response

2 The state equation can be solved by a synthesis of familiar techniques, as follows. First, we rearrange:

$$\frac{dx}{dt} - Ax = Bu. \quad (1)$$

An integrating factor would be clutch, but what should it be? It looks analogous to a scalar ODE that would use the natural exponential $\exp(-at)$ (for positive constant a), but we have a vector ODE. We need a matrix-version of the exponential. Recall that a series definition of the scalar exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is

We define the **matrix exponential** $\exp : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$ (we use the same symbol) to be, for $n \times n$ complex matrix Z,

$$\exp Z = \sum_{k=0}^{\infty} \frac{1}{k!} Z^k. \quad (2)$$

because why not? For the hell of it, let's see if the matrix exponential

$$\exp(-At) \quad (3)$$

works as an integrating factor, if for no other reason than it was constructed to be a sort of matrix-analog of $\exp(-at)$, which would work for the scalar case. Premultiplying (1) on both sides:

(exercise: prove $\mathrm{d}\exp(-At)/\mathrm{d}t = -\exp(-At)A$)

Rearranging and integrating over the interval $(0, t)$,

$(\exp(0) = I)$

This last expression can be solved for x, the **state response solution**. Before we do this, however, let's define the matrix function called the **state transition matrix** Φ to be the matrix-valued function

$$\Phi(t) = \exp(At), \tag{4}$$

Substituting Φ and solving,

$$\begin{aligned} x &= \Phi(t)\mathbf{x}(0) + \Phi(t)\int_0^t \Phi(-\tau)B\mathbf{u}(\tau)d\tau && \text{(5a)} \\ &= \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau. && \text{(5b)} \end{aligned}$$

Note that the first term of each version of Eq. 5 is the **free response** (due to initial conditions) and the second term is the **forced response** (due to inputs).

State transition matrix

3 The state transition matrix Φ introduced in Eq. 4 wound up being a key aspect of the response, which is why we call it that. We used two of its properties (in matrix exponential form) during that derivation: the

initial-value

$$\Phi(0) = I \qquad \text{(where I is the identity matrix)}$$

and the **inverse**

$$\Phi^{-1}(t) = \Phi(-t). \tag{6}$$

4 There is a third property that might be called the **bootstrapping property**: for time intervals Δt_i,

$$\Phi(\Delta t_1 + \Delta t_2 + \cdots) = \Phi(\Delta t_1)\Phi(\Delta t_2)\cdots. \tag{7}$$

This allows one to compute the state transition matrix[2] *incrementally*, from one previously computed.

5 A final property we'll consider is the special-case of a **diagonal** A with diagonal elements $a_{11}, a_{22}, \cdots, a_{nn}$, which yields a diagonal state transition matrix

6 The last property turns out to be quite convenient for **deriving** Φ for a given system, as we will see in Lec. 07.4 ssresp.diag. For now, we must rely on the definition of Φ from Eq. 4 and the series definition of the matrix exponential from Eq. 2. This requires us to derive the first several terms of the series solution and attempt to divine the corresponding scalar exponential series, a rather tedious task. Other than to familiarize ourselves with the definition through exercises, we prefer the derivation method of Lec. 07.4 ssresp.diag.

[2] As is common, we refer to it as the "state transition matrix at a certain time," but, technically, it's the *image* of the state transition matrix (which is actually a matrix-valued function) at a certain time. It is good to occasionally acknowledge the violence we do to math.

Output response

7 The output response $\mathbf{y}(t)$ requires little additional solution: assuming we have solved for the state response $\mathbf{x}(t)$, the output is given in the output equation Eq. 2b. Through direct substitution, we find the **output response solution**

$$\begin{aligned}\mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{u}(t) \qquad (8a)\\ &= C\Phi(t)\mathbf{x}(0) + C\int_0^t \Phi(t-\tau)B\mathbf{u}(\tau)d\tau + D\mathbf{u}(t). \qquad (8b)\end{aligned}$$