

07.1 `ssresp.response` Solving for the state-space response

1 In this lecture, we solve the state equation for the **state response** $\mathbf{x}(t)$ and substitute this into the output equation for the **output response** $\mathbf{y}(t)$.

State response

2 The state equation can be solved by a synthesis of familiar techniques, as follows. First, we rearrange:

$$\frac{d\mathbf{x}}{dt} - \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u}. \quad (1)$$

An integrating factor would be clutch, but what should it be? It looks analogous to a scalar ODE that would use the natural exponential $\exp(-at)$ (for positive constant a), but we have a vector ODE. We need a matrix-version of the exponential. Recall that a series definition of the scalar exponential function $\exp : \mathbb{C} \rightarrow \mathbb{C}$ is

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

We define the **matrix exponential** $\exp : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n \times \mathbb{C}^n$ (we use the same symbol) to be, for $n \times n$ complex matrix Z ,

$$\exp Z = \sum_{k=0}^{\infty} \frac{1}{k!} Z^k. \quad (2)$$

because why not? For the hell of it, let's see if the matrix exponential

$$\exp(-\mathbf{A}t) \quad (3)$$

works as an integrating factor, if for no other reason than it was constructed to be a sort of matrix-analog of $\exp(-at)$, which would work for the scalar case. Premultiplying (1) on both sides:

(exercise: prove $d \exp(-At)/dt = -\exp(-At)A$)

Rearranging and integrating over the interval $(0, t)$,

$$(\exp(0) = I)$$

This last expression can be solved for \mathbf{x} , the **state response solution**. Before we do this, however, let's define the matrix function called the **state transition matrix** Φ to be the matrix-valued function

$$\Phi(t) = \exp(At), \quad (4)$$

Substituting Φ and solving,

$$\mathbf{x} = \Phi(t)\mathbf{x}(0) + \Phi(t) \int_0^t \Phi(-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \quad (5a)$$

$$= \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t - \tau)\mathbf{B}\mathbf{u}(\tau)d\tau. \quad (5b)$$

Note that the first term of each version of Eq. 5 is the **free response** (due to initial conditions) and the second term is the **forced response** (due to inputs).

State transition matrix

3 The state transition matrix Φ introduced in Eq. 4 wound up being a key aspect of the response, which is why we call it that. We used two of its properties (in matrix exponential form) during that derivation: the

initial-value

$$\Phi(0) = I \quad (\text{where } I \text{ is the identity matrix})$$

and the **inverse**

$$\Phi^{-1}(t) = \Phi(-t). \quad (6)$$

4 There is a third property that might be called the **bootstrapping property**: for time intervals Δt_i ,

$$\Phi(\Delta t_1 + \Delta t_2 + \cdots) = \Phi(\Delta t_1)\Phi(\Delta t_2) \cdots. \quad (7)$$

This allows one to compute the state transition matrix² *incrementally*, from one previously computed.

5 A final property we'll consider is the special-case of a **diagonal** A with diagonal elements $a_{11}, a_{22}, \cdots, a_{nn}$, which yields a diagonal state transition matrix



6 The last property turns out to be quite convenient for **deriving** Φ for a given system, as we will see in [Lec. 07.4 ssresp.diag](#). For now, we must rely on the definition of Φ from [Eq. 4](#) and the series definition of the matrix exponential from [Eq. 2](#). This requires us to derive the first several terms of the series solution and attempt to divine the corresponding scalar exponential series, a rather tedious task. Other than to familiarize ourselves with the definition through exercises, we prefer the derivation method of [Lec. 07.4 ssresp.diag](#).

²As is common, we refer to it as the “state transition matrix at a certain time,” but, technically, it’s the *image* of the state transition matrix (which is actually a matrix-valued function) at a certain time. It is good to occasionally acknowledge the violence we do to math.

Output response

7 The output response $\mathbf{y}(t)$ requires little additional solution: assuming we have solved for the state response $\mathbf{x}(t)$, the output is given in the output equation [Eq. 2b](#). Through direct substitution, we find the **output response solution**

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{8a}$$

$$= \mathbf{C}\Phi(t)\mathbf{x}(0) + \mathbf{C} \int_0^t \Phi(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t). \tag{8b}$$